Submetido para TEMA

Log-Conformation Representation of Hiperbolic Conservation Laws with Source Term [1]

Abstract. The objective of this work is to study, through a simpler equation, the statement that the numerical instability associated to the high number of Weissenberg in equations with source term can be resolved by the use of the so called logarithmic representation conformation. We will focus on hyperbolic conservation laws, but more specifically on the advection equation with source term. The source term imposes a necessity of an elastic balance, as well as the CFL convective balance for stability. We will see that the representation of such equation by log-conformation removes the restriction of stability inherent to the elastic balance pointed out by [3] as the cause of high Weissenberg number problem (HWNP).

Key-words. source term, log-conformation representation (LCR), high Weissenberg number problem (HWNP).

1. Introduction

This assignment explores some important aspects of the numerical treatment, as well as the comparisons with the exact solution, of a simple partial differential equation, when to it is added a source term in analogy to what happens in the simulation of viscoelastic fluid flows [1],[3]. The search for robust techniques is still an important object of research, mainly because the most complete equations lead to new developments; this is the case of an equation with source term. A naive approach could lead one to believe that the equations without source and with source are alike and, therefore, every numerical method designed to the first would easily be applied to the second. In fact, this is not always the case. In a wider context, one case is the synthesized situation by the Weissenberg (Wi) number. In short, the bigger is Wi, the more intense the interactions of elastic nature and the more decisive is the source term contribution for the flow. Classical methods usually bump into restrictions regarding on the magnitude of Wi, about which is still to be discovered the cause: If it is of the numerical nature or, even, if intrinsically to the viscoelastic model itself. This situation has troubled, but also aroused the interest of many researchers in the last decade, constituting the so-called High Weissenberg Number Problem (HWNP). Recently, through the representation by

[1]Thanks to FAPESP, for the financial support; Paper presented at CMAC - Sudeste in 2013.

logarithmic conformation (LCR) there has been a better indication that the cause might be numerical [3]. In the same way that the numerical solution without source requires the correct convection balance, the solution with source needs to take into account a correct balance between the terms of convective nature (that generates the hyperbolic character) as well as those of elastic source (that introduces a stiff character in the equation).

On this assignment, we aim to effectively analyse and reproduce the mechanism associated to the HWNP through the study of a simplified equation. A simple equation that allows the broad study of these aspects is the advection equation with source.

Giving the hyperbolic equation in the conservative form, one can considered a source term so that the equation is still hyperbolic. However, not all extensions of numerical method designed for the first case are adequate to the new equation.

The advection equation is a hyperbolic equation, whose numerical solution must respect the CFL condition for the correct balance of convection [2],[5].

Let us consider the advection equation with source term

$$\underbrace{\frac{\partial u}{\partial t}}_{\text{transient term}} + \underbrace{a(x)\frac{\partial u}{\partial x}}_{\text{convective term}} = \underbrace{\left(b(x) - \frac{1}{Wi}\right)u,}_{\text{source term}} \tag{1.1}$$

where $u = u(x,t), x \in [0,L], t > 0, b = b(x) > 0, a = a(x)$ is the speed of advection and Wi is the Weissenberg number. This equation, also hyperbolic, when approximated by finite difference methods, as much as the balance imposed by convective CFL condition, requires an analysis for the elastic balance resulting from the addition of the source term.

The analytic solution of equation (1.1) can be deducted as

$$u(x,t) = \begin{cases} \exp(\frac{cx}{a}), & x \leq at \\ \exp(ct), & at < x \leq L \end{cases},$$

where $c = b(x) - \frac{1}{Wi}$.

We will see that the restriction of stability inherent to the elastic balance, pointed by [3] as the cause of the HWNP, when solving (1.1) can be removed by the representation by log-conformation.

2. Log-Conformation Representation

In order to clarify the instability problem mentioned above, we detailed and tested here the proposed methodology in [3]. Without loss of generality, in (1.1), we will consider $a(x) = a > 0$. Using in (1.1) the first-order Upwind Method (for the convective term) and Euler explicit (for the transient term), we obtain the scheme

$$\frac{U_{i,j+1} - U_{i,j}}{dt} = a\frac{U_{i,j} - U_{i-1,j}}{dx} = \left(b_i - \frac{1}{Wi}\right)U_{i,j} \tag{2.1}$$

and then

$$U_{i,j+1} = U_{i,j}\left[1 - \frac{a_i dt}{dx} + dt\left(b_i - \frac{1}{Wi}\right)\right] + \left(\frac{a_i dt}{dx}\right) U_{i-1,j}. \tag{2.2}$$

We consider $U_{i,j} = U(x_i, t_j)$ as being the solution provided by the numerical scheme in (x_i, t_j), with dx the spatial step, dt the time step and $b_i = b(x_i)$.

Then the numerical method (2.2) will be stable if

$$a\frac{dt}{dx} \leq 1 \tag{2.3}$$

and

$$1 - \frac{adt}{dx} + dt\left(b_i - \frac{1}{Wi}\right) \leq 1. \tag{2.4}$$

Observe that, for (2.4), it is enough

$$Wi < \frac{1}{b_i} \tag{2.5}$$

or

$$dx \leq \frac{a}{b_i - (Wi)^{-1}}. \tag{2.6}$$

Thus, besides the restriction CFL (2.3), we have (2.6) that is a restriction over the spatial step of the mesh, imposed by the elastic balance. Notice that (2.5) and (2.6) are affected by the Weissenberg number: in (2.6), we can see that, the bigger is Wi the smaller must be dx . On the other hand, (2.5) shows the direct relation between Wi e b. So, there is a maximum Wi permitted. These restrictions of stability are imposed by the HWNP.

Next, we will see that it is possible to remove the restriction (2.6) using the representation by log-conformation. That is based on the following statement:

The representation of a partial equation by log-conformation consists in replacing a range of unknowns in the partial equations by logarithmic unknowns. Thus, to represent a partial equation whose unknown is u, you must make the change of variable

$$\psi = log(u), \tag{2.7}$$

where

$$u = e^{\psi}. \tag{2.8}$$

Then, in (1.1), making the change of variable (2.8), we obtain

$$\frac{\partial\psi}{\partial t} + a(x)\frac{\partial\psi}{\partial x} = b(x) - \frac{1}{Wi}, \tag{2.9}$$

that corresponds to LCR version of equation (1.1).

Next, we discretized (2.9) by finite differences, considering $\Psi_{i,j} = \Psi(x_i, t_j)$ as being a solution provided by the numerical scheme at the point (x_i, t_j). So, approximating (2.9) by following (2.1), we obtain the following scheme:

$$\Psi_{i,j+1} = \Psi_{i,j}\left(1 - \frac{a_i dt}{dx}\right) + \Psi_{i-1,j}\left(\frac{a_i dt}{dx}\right) + dt\left(b_i - \frac{1}{Wi}\right). \tag{2.10}$$

Now, one can see that the numerical solution provided by such scheme will be stable when

$$\textbf{i)}\ a\frac{dt}{dx} \leq 1 \qquad \textbf{(ii)}\ 1 - a\frac{dt}{dx} \leq 1\ .$$

Then, the advection equation with source term represented by log-conformation does not impose restrictions for stability on the spatial step dx. Returning to the original variable, we have the less restrictive condition

$$dx \leq \frac{a}{log(b_i - 1/Wi)}. \tag{2.11}$$

3. Numerical Results

Next, we present numerical results obtained in implementing in MATLAB the exact solution of the equation (1.1), its numerical solution for (2.2), its LCR version (2.10) and also TVD schemes (Koren Limiter and Super Bee, cf.[4],[2]) in the convective term of the equation in question.

3.1. Influence of dt

Taking $a = 1, b = 2, Wi = 100, t = 2, L = 10, dx = 0.1$ and varying dt, the figures 1 and 2 represent the numerical solution of the equation (1.1) without LCR and with LCR. Comparing such pictures we can see that, regardless of the time step, the solution provided by the case with LCR (2.10) always follows the growth of the exact solution; however, as we decrease dt, the numerical solution provided by the method with LCR suffers a dissipation at the point of discontinuity of contact of the solution. In cases without LCR the numerical solution approximates the increasing of the exact solution as dt decreases. Table 1 shows the relative error (norm 2) made in each method while varying dt. Note that the error by norm 2, for being global, reflects the dissipation, but for the interval where the solution constant, the LCR is much more precise.

3.2. Influence of Wi

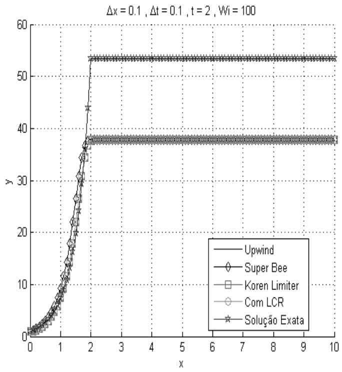
Taking $a = 1, b = 2, t = 2, L = 10, dx = 0.1$ and $dt = 0.1$, Table 2 presents the relative error made by the used numerical schemes when we take $Wi = 1, 2, 5, 10, 50, 100, 500$ e 1000. Figures 3 and 4 represent the numerical solution of the equation (1.1) when we take $Wi = 10, 100, 500$ e 1000; it is worth mentioning that, for other values of

Table 1: Errors for different dt.

dt	**CFL**	**ERRO**			
		LCR	**Upwind**	**Koren Limiter**	**Super Bee**
0.1	1	2.4506e-017	0.2901	0.2890	0.2890
0.05	0.5	0.0355	0.1674	0.1649	0.1644
0.01	0.1	0.0523	0.0552	0.0531	0.0526
0.001	0.01	0.0557	0.0429	0.0467	0.0468

Table 2: Errors for different Wi.

Wi	**ERRO**			
	LCR	**Upwind**	**Koren Limiter**	**Super Bee**
1	4.1252e-017	0.6960	0.6857	0.6857
2	3.7132e-017	0.1040	0.0848	0.0848
5	6.2840e-017	0.1741	0.1701	0.1701
10	2.3574e-017	0.2370	0.2350	0.2350
50	2.3941e-017	0.2844	0.2832	0.2832
100	2.4506e-017	0.2901	0.2890	0.2890
500	6.4126e-017	0.2947	0.2936	0.2936
1000	6.8731e-017	0.2952	0.2941	0.2941

a) 

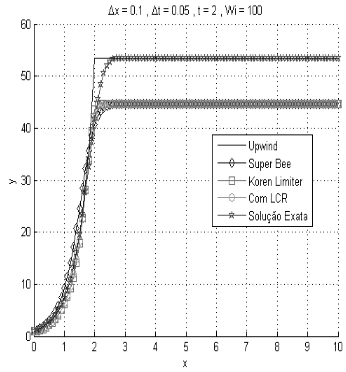
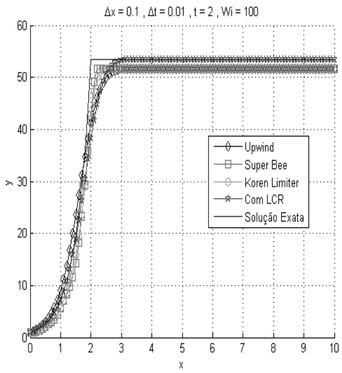
b) 

Figure 1: a) dt = 0,1; b) dt = 0,05

a) 

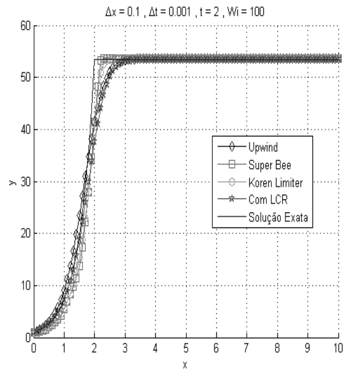
b) 

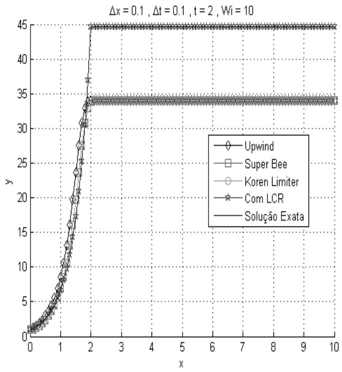
Figure 2: a) dt = 0,01; b) dt = 0,001

Wi considered, the pictures obtained are all "similar". In every case the numerical approximation increase with LCR (2.10) proved to be more precise.

In table 2 we can see that regardless the considered Wi , the relative error of the method with LCR is always smaller than the relative error of the methods without LCR.

4. Conclusion

Considering discretization by first order upwind in the convective term and Euler explicit in the term transient, we note that when including a source term in the advection equation it is also "inserted" a restriction of stability (2.6) to the spatial mesh and that such restriction is influenced by the Weissenberg number. Thus, the higher Wi, much more refined must be the spatial mesh. This situation is related



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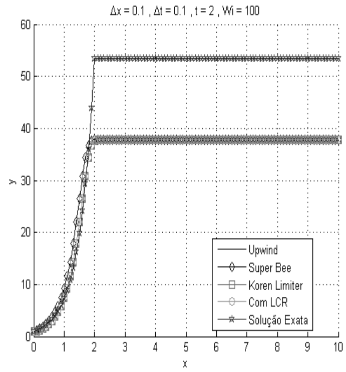

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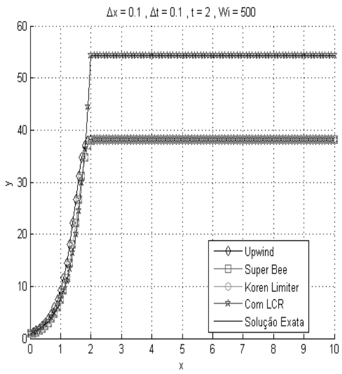
Figure 3: a) We = 10; b) We = 100



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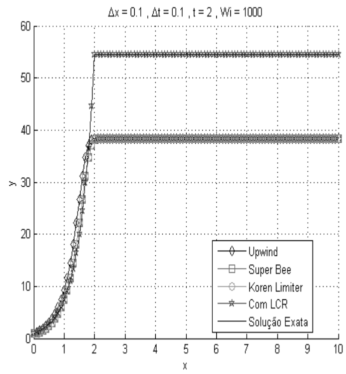

b)

Figure 4: a) We = 500; b) We = 1000

to the HWNP.

Corroborating with [3] and showing in a very concrete way, through a simple problem with exact solution, we can see that there is a strong "damping" for the numerical solution without LCR, but also that such a situation can be improved by a change of variable in the partial equation, characterizing the called representation by logarithmic conformation of the partial equation, so that there is less restriction for stability allowing an adequate "exponential" increasing of the numerical solution in order to go along with the exact solution. Numerical results verified these assumptions.

References

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