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Finite Difference Preserving the Energy Properties of a Coupled System of Diffusion Equations

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Abstract. In this paper we proved the exponential decay of the energy of a numerical scheme in finite difference applied to a coupled system of diffusion equations. At the continuous level, it is well-known that the energy is decreasing and stable in the exponential sense. We present in detail the numerical analysis of exponential decay to numerical energy since holds the stability criterion.

Keywords: diffusion equations; finite difference; numerical exponential decay.

1. Introduction

In this work we consider the numerical solutions in finite difference applied to the following coupled system of diffusion equations:

$$\begin{aligned}
\phi_t - D_1\phi_{xx} + \alpha(\phi - \psi) &= 0, \quad \Omega \times (0,T), && (1.1)\\
\psi_t - D_2\psi_{xx} + \alpha(\psi - \phi) &= 0, \quad \Omega \times (0,T), && (1.2)\\
\phi(0,t) = \phi(L,t) = \psi(0,t) = \psi(L,t) &= 0, \quad t > 0, && (1.3)\\
\phi(x,0) = \phi_0(x), \quad \psi(x,0) &= \psi_0(x), \ \forall x \in \Omega, && (1.4)
\end{aligned}$$

where $\Omega = (0, L)$, $D_1 > 0$ and $D_2 > 0$ are the diffusion coefficients and $\alpha > 0$ is the coupling parameter. The initial-boundary value problem $(1.1) - (1.4)$ appears in dispersion processes between species. To more detail see Murray [5].

An important non-linear functional concerning to the system system $(1.1)-(1.4)$ is its energy. It is given by

$$\mathcal{E}(t) := \frac{1}{2}\int_0^L \left[|\phi(x,t)|^2 + |\psi(x,t)|^2\right] dx, \quad \forall t \geq 0. \tag{1.5}$$

Naturally, in diffusion problems, one has

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$$\mathcal{E}(t) \leq \mathcal{E}(0), \ \forall t \geq 0. \tag{1.6}$$

Moreover, the energy (1.5) suggests that the system $(1.1)-(1.4)$ is well-posed in Hilbert space $\mathcal{H} = L^2(0,L) \times L^2(0,L)$. That is to say, for initial data $(\phi_0, \psi_0) \in \mathcal{H}$ there exists a unique solution $(\phi, \psi) \in C([0,T]; L^2(0,L)) \cap C^1([0,T]; L^2(0,L))$. Then, the existence and uniqueness of solutions can be assured by using the semigroups theory (see [4]).

Another important property concerning to the energy $\mathcal{E}(t)$ is its decay to zero as $t \to \infty$ (see Murray [5]). In that direction, we focus on numerical analysis of the energy properties of a numerical scheme in finite difference applied to the system $(1.1) - (1.4)$.

The rest of the paper is organized as follows: In section 2, we proved the energy dissipation property. In section 3, we derived the numerical scheme in finite difference and its numerical energy. In particular, we showed that this numerical energy preserves the exponential decay such as continuous case. In section 4, we present some numerical results. In section 5, we finished with the conclusions.

2. Energy properties

In this section, we showed that $\mathcal{E}(t)$ obeys the energy dissipation law. Indeed, we have the following Theorem:

Theorem 2..1 (Energy dissipation). *The energy $\mathcal{E}(t)$ in (1.5) satisfies the energy dissipation law. More precisely, one has*

$$\mathcal{E}(t) \leq \mathcal{E}(0), \ \forall t \geq 0.$$

Demonstração. Multiplying (1.1) by ϕ and integrating on $(0,L)$, we get

$$\int_0^L \phi_t \phi dx - D_1 \int_0^L \phi_{xx} \phi dx + \alpha \int_0^L (\phi - \psi)\phi dx \quad = \quad 0. \tag{2.7}$$

Taking into account the homogeneous Dirichlet boundary conditions (1.3), it follows that

$$\frac{1}{2}\frac{d}{dt}\int_0^L |\phi|^2 dx + D_1 \int_0^L |\phi_x|^2 dx + \alpha \int_0^L (\phi - \psi)\phi dx \quad = \quad 0. \tag{2.8}$$

Analogously, we obtain

$$\frac{1}{2}\frac{d}{dt}\int_0^L |\psi|^2 dx + D_2 \int_0^L |\psi_x|^2 dx + \alpha \int_0^L (\psi - \phi)\psi dx \quad = \quad 0. \tag{2.9}$$

Adding up (2.8) and (2.9), we obtain

$$\frac{1}{2}\frac{d}{dt}\int_0^L \left[|\phi|^2 + |\psi|^2\right]dx = -\int_0^L \left[D_1|\phi_x|^2 + D_2|\psi_x|^2 + \alpha|\phi-\psi|^2\right]dx, \quad (2.10)$$

from where we have

$$\frac{d}{dt}\mathcal{E}(t) \leq 0, \ \forall t \geq 0,$$

since $\alpha, D_i, i = 1, 2,$. Therefore,

$$\mathcal{E}(t) \leq \mathcal{E}(0), \ \forall t \geq 0.$$

□

Theorem (2..1) shows that $\mathcal{E}(t)$ is decreasing along time t. Moreover, to questions on exponential decay, we refer the readers to Murray [5]. Next, we will apply the finite difference method to the system (1.1)-(1.4).

3. Numerical Formulation in Finite Difference

In this section, we use the standard finite difference method to analyze the qualitative properties of system (1.1)-(1.4).

Given $J, N \in \mathbb{N}$ we set $h = \Delta x = \dfrac{L}{J+1}$ and $\Delta t = \dfrac{T}{N+1}$ and we introduce the nets

$$\begin{aligned} 0 &= x_0 < x_1 < ... < x_j = j\Delta x < ... < x_J < x_{J+1} = L, \quad (3.1)\\ 0 &= t_0 < t_1 < ... < t_n = n\Delta t < ... < t_N < t_{N+1} = T, \quad (3.2)\end{aligned}$$

where $x_j = j\Delta x$ and $t_n = n\Delta t$ for $j = 0, 1, 2, ..., J+1$ and $n = 0, 1, 2, ..., N+1$.

We assume the standard numerical operators applied to diffusion problems. We use the following operators in finite difference to the function u:

$$\partial_x\overline{\partial}_x u_j^n := \frac{u_{j+1}^n - u_j^n + u_{j-1}^n}{\Delta x^2}, \quad \partial_t u_j^n := \frac{u_j^{n+1} - u_j^n}{\Delta t}. \quad (3.3)$$

Our problem consists in to find ϕ_j^n e ψ_j^n satisfying

$$\begin{aligned} \partial_t\phi_j^n - D_1\overline{\partial}_x\partial_x\phi_j^n + \alpha(\phi_j^n - \psi_j^n) &= 0, \ \forall j, \ \ 1 \leq j \leq J \quad (3.4)\\ \partial_t\psi_j^n - D_2\overline{\partial}_x\partial_x\psi_j^n + \alpha(\psi_j^n - \phi_j^n) &= 0, \ \forall j, \ \ 1 \leq j \leq J \quad (3.5)\\ \phi_0^n = \phi_{J+1}^n = 0, \psi_0^n = \psi_{J+1}^n = 0 &= 0, \ \forall n, \ \ 0 \leq n \leq N, \quad (3.6)\\ \phi_j^0 = \phi(x_j, 0), \ \psi_j^0 &= \psi(x_j, 0) \ \forall j, \ \ 0 \leq j \leq J. \quad (3.7)\end{aligned}$$

The choice of forward Euler relies in the questions of numerical stability. Indeed, it is well-known that the numerical solutions obtained for classical diffusion equations (single diffusion equation) converge if $r := \Delta t/\Delta x^2 \leq 1/2$.

It is easy to see that system (3.4)–(3.7) is consistent and the truncation error is $\mathcal{O}(\Delta t, \Delta x^2)$. Being consistent and stable, it follows by Theorem of Lax [2, 3, 6] the respective convergence. Next, we use energy arguments to identify the stability criterion to the system (3.4)–(3.7).

3.1. Numerical Energy

In this section, we obtain the numerical energy concerning to the numerical equations (3.4) – (3.7). We will show that it is given by

$$\mathcal{E}^n := \Delta x \sum_{j=0}^{J} \left[|\phi_j^n|^2 + |\psi_j^n|^2 \right], \ \forall n = 0, 1, ..., N, N+1. \tag{3.8}$$

Note that $\mathcal{E}^n$ is composed by norms in the space l^2. Moreover, it is the numerical counterpart of energy in (1.5). In Theorem we used the energy method to show that the solutions of (1.1) – (1.4) are limited by initial data. An analysis similar can be made to numerical solutions of (3.4)-(3.7).

For the sake of simplicity, we assume the particular case $D_1 = D_2 = 1$. Making this, we obtain two systems from (3.4)-(3.7). The first of them is given by

$$\partial_t \omega_j^n - \overline{\partial}_x \partial_x \omega_j^n = 0, \qquad \forall j, \ \ 1 \leq j \leq J \tag{3.9}$$

$$\omega_0^n = \omega_{J+1}^n = 0, \qquad \forall n, \ \ 0 \leq n \leq N \tag{3.10}$$

$$\omega_j^0 = \omega(x_j, 0), \qquad \forall j, \ \ 0 \leq j \leq J, \tag{3.11}$$

and the other one is

$$\partial_t \theta_j^n - \overline{\partial}_x \partial_x \theta_j^n + 2\alpha \theta_j^n = 0, \qquad \forall j, \ \ 1 \leq j \leq J \tag{3.12}$$

$$\theta_0^n = \theta_{J+1}^n = 0, \qquad \forall n, \ \ 0 \leq n \leq N \tag{3.13}$$

$$\theta_j^0 = \theta(x_j, 0), \qquad \forall j, \ \ 0 \leq j \leq J, \tag{3.14}$$

where $\omega_j^n = \phi_j^n + \psi_j^n$ and $\theta_j^n = \phi_j^n - \psi_j^n$. The energies are given by

$$E^n := \Delta x \sum_{j=0}^{J} |\omega_j^n|^2 \ \text{ and } \ \widetilde{E}^n := \Delta x \sum_{j=0}^{J} |\theta_j^n|^2. \tag{3.15}$$

respectively. Then, we have

$$\mathcal{E}^n := \frac{E^n + \widetilde{E}^n}{2}, \ \forall n = 0, 1, ..., N, N+1. \tag{3.16}$$

Our first result concerning to the numerical energy is given by following Theorem:

Theorem 3..1. *For* $1-2r-2\alpha\Delta t \geq 0$ *where* $r := \Delta t/\Delta x^2 \leq 1/2$, *holds*

$$\mathcal{E}^n \leq \mathcal{E}^0, \ \forall\ n = 1, 2, ..., N, N+1.$$

Demonstração. First, we note that

$$\widetilde{E}^{n+1} - \widetilde{E}^{n} = \Delta x \sum_{j=0}^{J} (|\theta_j^{n+1}|^2 - |\theta_j^n|^2) = \Delta x \sum_{j=0}^{J} (\theta_j^{n+1} + \theta_j^n)(\theta_j^{n+1} - \theta_j^n), \quad (3.17)$$

and for $r = \Delta t/\Delta x^2$ we obtain from (3.12)

$$\theta_j^{n+1} - \theta_j^n = r(\theta_{j+1}^n - 2\theta_j^n + \theta_{j-1}^n) - 2\alpha\Delta t\theta_j^n. \quad (3.18)$$

Substituting (3.18) into (3.17) we obtain

$$\begin{aligned}
\widetilde{E}^{n+1} - \widetilde{E}^{n} &= \Delta x \sum_{j=0}^{J} (\theta_j^{n+1} + \theta_j^n)\Big[r(\theta_{j+1}^n - 2\theta_j^n + \theta_{j-1}^n) - 2\alpha\Delta t\theta_j^n\Big] \\
&= r\Delta x \sum_{j=0}^{J} (\theta_j^{n+1} + \theta_j^n)(\theta_{j+1}^n - 2\theta_j^n + \theta_{j-1}^n) - 2\alpha\Delta t\Delta x \sum_{j=0}^{J} \theta_j^n(\theta_j^{n+1} + \theta_j^n) \\
&= r\Delta x \sum_{j=0}^{J} \theta_j^n(\theta_{j+1}^n - 2\theta_j^n + \theta_{j-1}^n) + r\Delta x \sum_{j=0}^{J} \theta_j^{n+1}(\theta_{j+1}^n + \theta_{j-1}^n) \\
&- 2(r+\alpha\Delta t)\Delta x \sum_{j=0}^{J} \theta_j^{n+1}\theta_j^n - 2\alpha\Delta t\Delta x \sum_{j=0}^{J} |\theta_j^n|^2. \quad (3.19)
\end{aligned}$$

Now, having in mind the homogeneous Dirichlet boundary conditions (3.13), we have the following identities:

$$\begin{aligned}
\sum_{j=0}^{J} \theta_j^n(\theta_{j+1}^n - 2\theta_j^n + \theta_{j-1}^n) &= -\sum_{j=0}^{J} |\theta_{j+1}^n - \theta_j^n|^2; \quad (3.20) \\
\sum_{j=0}^{J} \theta_j^{n+1}\theta_j^n &= \sum_{j=0}^{J} \Big[\theta_j^n + r(\theta_{j+1}^n - 2\theta_j^n + \theta_{j-1}^n) - 2\alpha\Delta t\theta_j^n\Big]\theta_j^n \\
&= \sum_{j=0}^{J} \Big[|\theta_j^n|^2 + r\theta_j^n(\theta_{j+1}^n - 2\theta_j^n + \theta_{j-1}^n) - 2\alpha\Delta t|\theta_j^n|^2\Big] \\
&= (1-2\alpha\Delta t)\sum_{j=0}^{J} |\theta_j^n|^2 - r\sum_{j=0}^{J} |\theta_{j+1}^n - \theta_j^n|^2. \quad (3.21)
\end{aligned}$$

Moreover, taking into account the inequality $ab \le (a^2+b^2)/2$, we have

$$\begin{aligned} \sum_{j=0}^{J} \theta_j^{n+1}(\theta_{j+1}^n + \theta_{j-1}^n) &\le \frac{1}{2}\sum_{j=0}^{J}(|\theta_j^{n+1}|^2 + |\theta_{j+1}^n + \theta_{j-1}^n|^2) \\ &\le \sum_{j=0}^{J}(|\theta_j^{n+1}|^2 + |\theta_j^n|^2). \end{aligned} \tag{3.22}$$

Substituting (3.20) – (3.22) into (3.19), we obtain

$$\begin{aligned} \widetilde{E}^{n+1} - \widetilde{E}^n &\le -r\Delta x \sum_{j=0}^{J} |\theta_{j+1}^n - \theta_j^n|^2 + r\Delta x \sum_{j=0}^{J}(|\theta_j^{n+1}|^2 + |\theta_j^n|^2) \\ &- 2(r+\alpha\Delta t)(1-2\alpha\Delta t)\Delta x \sum_{j=0}^{J}|\theta_j^n|^2 - 2\alpha\Delta t\Delta x\sum_{j=0}^{J}|\theta_j^n|^2 \\ &+ r(2r + 2\alpha\Delta t)\Delta x \sum_{j=0}^{J}|\theta_{j+1}^n - \theta_j^n|^2, \end{aligned}$$

and then

$$\begin{aligned} (1-r)(\widetilde{E}^{n+1} - \widetilde{E}^n) &\le -4\alpha\Delta t(1 - r - \alpha\Delta t)\widetilde{E}^n \\ &- r(1 - 2r - 2\alpha\Delta t)\Delta x\sum_{j=0}^{J}|\theta_{j+1}^n - \theta_j^n|^2. \end{aligned}$$

Taking $1 - 2r - 2\alpha\Delta t \ge 0$ we choose $\gamma := 1 - r - \alpha\Delta t \ge 0$ and then

$$(1-r)(\widetilde{E}^{n+1} - \widetilde{E}^n) \le -4\alpha\gamma\Delta t\widetilde{E}^n \quad \Rightarrow \quad \widetilde{E}^{n+1} \le \left(1 - \frac{4\alpha\gamma\Delta t}{1-r}\right)\widetilde{E}^n,$$

since $r < 1$. Therefore

$$\widetilde{E}^n \le \left(1 - \frac{4\alpha\gamma\Delta t}{1-r}\right)^n \widetilde{E}^0, \quad n = 0, 1, 2, ..., N, N+1, \tag{3.23}$$

assuring that $\widetilde{E}^n$ is decreasing for $\gamma = 1 - r - \alpha\Delta t \ge 0$. Finally, we can written

$$2\mathcal{E}^n = \widetilde{E}^n + E^n \le \left(1 - \frac{4\alpha\gamma\Delta t}{1-r}\right)^n \widetilde{E}^0 + E^0 \le \widetilde{E}^0 + E^0 \le 2\mathcal{E}^0,$$

from where we have

$$\mathcal{E}^n \leq \mathcal{E}^0, \quad \forall\, n = 1, 2, ..., N, N+1.$$

$\square$

3.2. Uniform Exponential Decay

In this section, we present a result on exponential decay to the energy of the system $(3.4)-(3.7)$. We assure that numerical scheme preserves the important property on exponential decay such as continuous case. Here, we consider the case $D_1 = D_2 = 1$.

Theorem 3..2. *If* $1-2r-2\alpha\Delta t \geq 0$ *then* $\mathcal{E}^n$ *is exponentially stable. More precisely,*

$$\mathcal{E}^n \leq \mathcal{E}^0 e^{-\left[\frac{8}{\Delta x^2}\sin^2\left(\frac{\pi\Delta x}{2}\right)\Delta t\right]n}. \tag{3.24}$$

Demonstração. We consider the uncoupled systems $(3.9)-(3.11)$ and $(3.12)-(3.14)$. The first of them is exponentially stable, i.e.,

$$E^n \leq E^0 e^{-2n\lambda_1\Delta t}, \tag{3.25}$$

where $\lambda_1 = \dfrac{4}{\Delta x^2}sen^2\left(\dfrac{\pi\Delta x}{2}\right)$ (see [1]).

For the other one, we assume the decomposition given by $\theta_j^n = X_j T^n$, $j = 1, 2, ..., J+1$, $n \geq 0$ and we obtain

$$\theta_j^n = [1 - (\lambda_k + 2\alpha)\Delta t]^n sen(k\pi x_j), \quad \forall\, k, j = 1, 2, ..., J+1, \tag{3.26}$$

where $\lambda_k = \dfrac{4}{\Delta x^2}sen^2\left(\dfrac{k\pi\Delta x}{2}\right)$, $k = 1, 2, ..., J$.

Now, it follows immediately that

$$\widetilde{E}^n = |1 - (\lambda_k + 2\alpha)\Delta t|^{2n}\Delta x\sum_{j=0}^{J} sen^2(k\pi x_j) = \frac{\pi}{2}|1 - (\lambda_k + 2\alpha)\Delta t|^{2n},$$

and then

$$\widetilde{E}^n = \widetilde{E}^0|1 - (\lambda_k + 2\alpha)\Delta t|^{2n} \leq \widetilde{E}^0|1 - (\lambda_1 + 2\alpha)\Delta t|^{2n} \approx \widetilde{E}^0 e^{-2n(\lambda_1+2\alpha)\Delta t}, \tag{3.27}$$

since $1 - 2r - 2\alpha\Delta t \geq 0$. Finally, to obtain the estimative of exponential decay, we take the inequalities (3.25) and (3.27), from where we have

$$
\begin{aligned}
2\mathcal{E}^n &= E^n + \widetilde{E}^n \leq e^{-2n\lambda_1\Delta t}E^0 + e^{-2n\Delta t(\lambda_1+2\alpha)}\widetilde{E}^0 \\
&\leq (E^0 + \widetilde{E}^0)e^{-2n\lambda_1\Delta t} = 2\mathcal{E}^0 e^{-2n\lambda_1\Delta t}.
\end{aligned}
$$

Therefore, we obtain

$$
\mathcal{E}^n \leq \mathcal{E}^0 e^{-2n\lambda_1\Delta t}.
$$

□

4. Numerical Experiments

In this section, we present numerical results obtained with the scheme $(3.4)-(3.7)$. We use the following data: $L = 1,\ T = 0.045s$ and 20 divisions in space and 38 in time. To initial data we use $\phi(x_j, 0) = \sin(\pi x_j)$, $\psi(x_j, 0) = 3\sin(3\pi x_j)$. According stability restrictions given by Theorems (3.1) and (3.2) result $\alpha < 0,444 \cdot 10^3$.

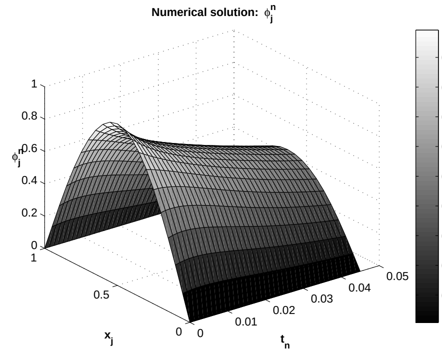

Figura 1: $\alpha = 0,444 \cdot 10^1$

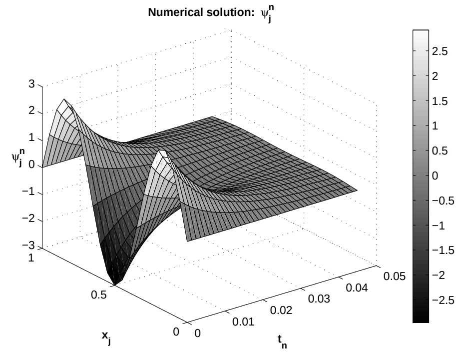

Figura 2: $\alpha = 0,444 \cdot 10^1$

In figures 1 and 2 we see the correct behavior of the solutions ϕ_j^n and ψ_j^n. That is to say, they are increasing and free of numerical oscillations because the stability criterion is obeys. On the other hand, significative changes are showed in figures 3 and 4 for α takes of order 10^3.

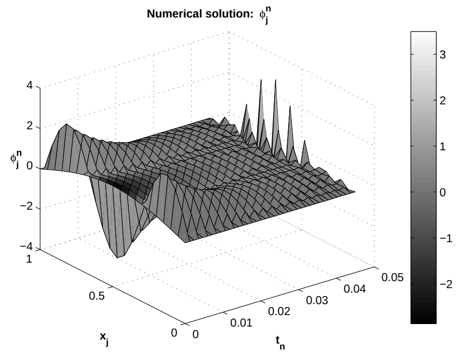

Figura 3: $\alpha = 0,765 \cdot 10^3$

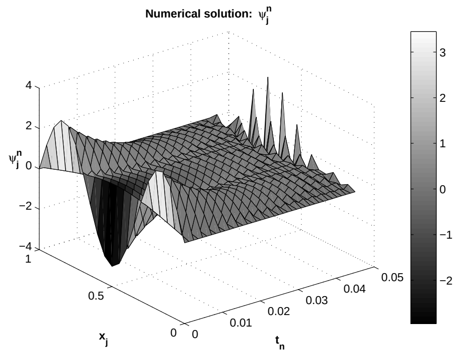

Figura 4: $\alpha = 0,765 \cdot 10^3$

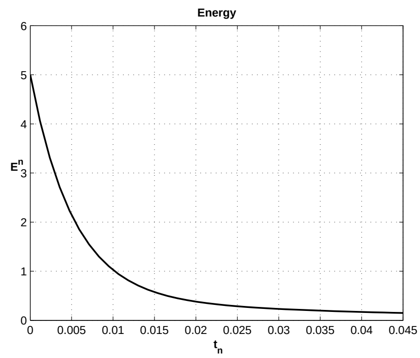

Figura 5: $\alpha = 0,444 \cdot 10^1$

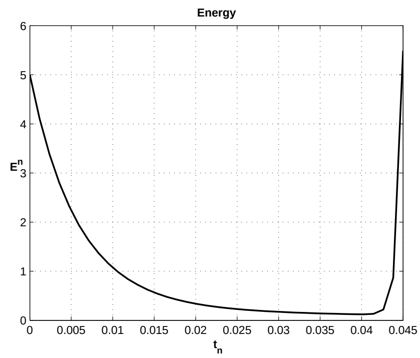

Figura 6: $\alpha = 0,765 \cdot 10^3$

Finally, looking to the figure 5, we note that $\mathcal{E}^n$ is stable in the exponential sense since that stability criterion (see Theorem 3..2) prevails. Otherwise, numerical instabilities occur (see figure 6).

5. Conclusion

In this paper we showed in detail the numerical analysis of exponential decay of the solutions of a numerical scheme in finite difference applied to coupled systems of diffusion equations. This scheme preserves the exponential decay since holds the stability criterion. We illustrated this property by means of some numerical experiments. Another numerical schemes can be used in order to obtain the exponential decay. For example, it is well-known that implicit schemes are free of stability criterion (unconditionally stable). Therefore, implicit schemes can be used in this context to preserve the exponential decay.

Resumo.

Neste trabalho, nós provamos a propriedade de decaimento exponencial da energia numérica associada a um particular esquema numérico em diferenças finitas aplicado a um sistema acoplado de equações de difusão. Ao nível da dinâmica do contínuo, é bem conhecido que a energia do sistema é decrescente e exponencialmente estável. Aqui nós apresentamos em detalhes a análise numérica de decaimento exponencial da energia numérica desde que obedecido o critério de estabilidade.

Palavras-chave: equações de difusão; diferenças finitas; decaimento exponencial numérico.

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