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Robustness on Intuitionistic Fuzzy Connectives*

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ABSTRACT. The main contribution of this paper is concerned with the robustness of intuitionistic fuzzy connectives in fuzzy reasoning. Starting with an evaluation of the sensitivity in n-order functions on the class of intuitionistic fuzzy sets, we apply the results in the intuitionistic (S, N)-implication class. The paper formally states that the robustness preserves the projection functions in this class.

Keywords: robustness, δ-sensitivity, perturbation, fuzzy logic, intuitionistic fuzzy logic, intuitionistic fuzzy connectives.

1 INTRODUCTION

Robustness or sensitivity can be conceived as a fundamental property of a logical system stating that the conclusions are not essentially changed if the assumed conditions varied within reasonable parameters. It is a relevant research area with important contributions [30, 32, 23, 33, 18].

This paper considers the δ-sensitivity (or pointwise senstivity) study as presented in [19] but related to intuitionistic fuzzy connectives (IFCs), providing logical foundations to support robust fuzzy applications based on the Atanassov's Intuitionistic Fuzzy Logic [1, 4] (IFL). Additionally, the δ-sensitivity of inference systems based on IFL can be analogously studied in the concepts of multi-valued fuzzy logic – the Interval-valued Atanassov's Fuzzy Logic [12] and integrated to challenging approach concerned with the Interval-valued Intuitionistic Fuzzy Logic [3].

Thus, in systems based on fuzzy rules, each linguistic term of an input linguistic variable is associated with a given fuzzy set [31]. Since the definition of these fuzzy sets is highly subjective, such fuzzy system should be stable in the sense that smooth changes performed on the input fuzzy sets should result only a slight change on their outputs. So, an immediate question follow as: how does one can ensure stability of such systems by applying Fuzzy Logic (FL) and the corresponding intuitionistic extension?

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As a contribution to elucidate this question, this paper considers the robustness analysis defined on δ-sensitivity and related to Atanassov's intuitionistic fuzzy connectives. Such analysis can improve the stability study of systems based on Atanassov's intuitionistic fuzzy rules.

1.1 Relevance of the robustness analysis based on δ-sensitivity

In this paper we are analysing the δ-sensitivity of the steps of the fuzzy-rule inference engine, dealing with logical connectives as algebraic n-order function, meaning that desirable logical and intuitionistic properties are described by algebraic properties.

The main idea related to the δ-sensitivity study of an IFC was to provide an logical approach as foundation to applied situations where the designer does not have complete knowledge about of linguistic variables modelling the relationship of membership and non-membership functions involved in an application.

Such proposed logical approach is performed by a $\Delta_f(\mathbf{x}, \delta)$ operator related to n-order function $f : U^n \to U$, taking into account an input $\mathbf{x}$ and a δ-parameter. Thus, it provides an interpretation of a function disturbance of fuzzy connective (FC), which is modelled by a δ-parameter and a function f taking an input $\mathbf{x}$, respectively. Indeed, a study on the influence and sensitivity of such δ-parameter in a knowledge base leads to an improvement in the fuzzy system (faced on, e.g., variances related to noise or temporal devices).

Thus, as FCs (mainly negations, t-norms and t-conorms, implications and coimplications) are important elements in the fuzzy reasoning, the corresponding investigation of the δ-sensitivity related to the pointwise analysis on the arguments of such operators, in terms of [19] and [25], will be carried out in this work.

Firstly, this paper investigates how to measure the robustness of IFCs using the sensitivity of FCs and it also derives the best perturbation parameters of intuitionistic fuzzy reasoning. Based on these results, an extension of such approach to N-dual constructions of IFCs is performed.

1.2 Applications of the robustness analysis in Fuzzy Logic

In the research area of fuzzy control, one of the most important problems is the analysis of stability and robustness of fuzzy controllers [17].

Significant works have been developed in this research area of fuzzy control, whose main research problem is related to the analysis of stability and robustness of fuzzy controllers, e.g. [10, 20] and [32]. In [30], the concepts of maximum and average robustness of fuzzy sets were already proposed.

Sensitivity analysis has become a major tool in the assessment of the reliability of engineering structures. Given an input-output system, the question is which input variables have the most decisive influence on the output on such systems. In [23], methods of modelling correlations and interactivity in such systems are investigated. In [18] the fundamental property of robustness of interval-valued fuzzy inference is studied. Moreover, robust interval matrices over (max, min)-

algebra (fuzzy matrices) are studied and equivalent conditions for interval fuzzy matrices to be robust are presented in [22].

The notion of δ-equalities of fuzzy sets is used in [10] to study robustness of fuzzy reasoning based on fuzzy implication operators, generalized modus ponens and generalized modus tollens. Other relations among the robustness of fuzzy reasoning, fuzzy conjunctions and classes of implication operators were presented in [17].

More recently, the robustness of fuzzy reasoning from the perspective of perturbation of membership functions is considered in [21] also including a method for judging the most robust elements of different classes of fuzzy connectives. Additionally, a new method for sensitivity analysis of fuzzy transportation problems is proposed in [8].

1.3 Main contribution and paper outline

This paper considers the notion of δ-sensitivity of fuzzy connectives in the Atanassov's intuitionistic fuzzy approach [2], which is characterized by the non-complementary relationship between the membership and non-membership functions.

Since δ-sensitivity on interpretation of IFCs is closed related to truth and non-truth in conditional fuzzy rules, this work is focused not only in the representable Atanassov's intuitionistic fuzzy t-norms and implications but also in their corresponding dual fuzzy connectives which are representable intuitionistic fuzzy t-conorms and coimplications.

As the main result, the δ-sensitivity of the (S, N)-intuitionistic fuzzy implication class is introduced, based on the study of δ-sensitivity of both classes, the intuitionistic fuzzy negations and t-conorms. Moreover, the paper is extending the work in [26] to the dual Atanassov's intuitionistic approach.

The preliminaries describes the basic concepts of FCs and IFCs. The δ-sensitivity of FCs and general results of robustness of IFCs are stated in Sections 3 and 4, respectively. Final remarks are reported in the conclusion.

2 PRELIMINARIES

We start by recalling some basic concepts of FCs and IFCs we are going to use in our subsequent developments.

2.1 Fuzzy connectives

Firstly, notions concerning t-(co)norms, (co)implications and dual functions are reported based on [15] and [16].

2.1.1 Fuzzy negations

Let $U = [0, 1]$ be the unit interval of real numbers. Recall that a function $N : U \to U$ is a **fuzzy negation** if it satisfies the properties:

N1: $N(0) = 1$ and $N(1) = 0$; **N2**: If $x \geq y$ then $N(x) \leq N(y)$, $\forall x, y \in U$.

A fuzzy negation satisfying the involutive property:

N3: $N(N(x)) = x, \forall x \in U$,

is called a **strong fuzzy negation** (SFN), e.g. the standard negation $N_S(x) = 1 - x$. When $\mathbf{x} = (x_1, x_2, \ldots, x_n) \in U^n$ and N is a fuzzy negation, the following notation is considered:

$$N(\mathbf{x}) = (N(x_1), N(x_2), \ldots, N(x_n)) \tag{2.1}$$

Let N be a negation. The N**-dual function** of $f : U^n \to U$ is given by:

$$f_N(\mathbf{x}) = N(f(N(\mathbf{x}))), \quad \forall \mathbf{x} \in U^n. \tag{2.2}$$

Notice that, when N is involutive, $(f_N)_N = f$, that is the N-dual of f_N is the function f. In addition, a function f for which $f = f_N$ is called self-dual function.

2.1.2 Triangular norms and conorms

A function $T : U^2 \to U$ is a **triangular-norm** (t-norm) if and only if it satisfies, for all $x \in U$, the following properties.

T1: $T(x, 1) = x$;

T2: $T(x, y) = T(y, x)$;

T3: $T(x, T(y, z)) = T(T(x, y), z)$;

T4: if $x \leq x'$, $T(x, y) \leq T(x', y)$.

The notion of a triangular conorm (t-conorm) $S : U^2 \to U$ can be defined in the same manner with the exception that the identity **T1** should be replaced by **S1**: $S(0, x) = x$, for all $x \in U$.

Let N be a fuzzy negation on U. The mappings $T_N, S_N : U^2 \to U$ denoting the N-dual functions of a t-norm T and a t-conorm S, respectively, are defined as:

$$T_N(x, y) = N(T(N(x), N(y))), \qquad S_N(x, y) = N(S(N(x), N(y))). \tag{2.3}$$

2.1.3 Fuzzy implications and coimplications

An **implicator operator** $I : U^2 \to U$ extends the classical implication function:

I0: $I(1, 1) = I(0, 1) = I(0, 0) = 1, \; I(1, 0) = 0.$

Definition 1. *[16] When $x, y, z \in U^2$, a fuzzy implication $(J)I : U^2 \to U$ is an implicator verifying the properties from **I1** to **I4** described in the following:*

I1: $I(x, y) \geq I(z, y)$ *if* $x \leq z$ *(first place antitonicity);*

I2: $I(x, y) \leq I(x, z)$ *if* $y \leq z$ *(second place isotonicity);*

I3: $I(0, y) = 1$ *(dominance of falsity);*

I4: $I(x, 1) = 1$ *(boundary condition);*

Analogously, the notion of a coimplicator $J : U^2 \to U$ can be defined as an extension of the classical coimplication function. Thus, such operators satisfy the corresponding boundary conditions:

J0: $J(0, 0) = J(1, 0) = J(1, 1) = 0,\ J(0, 1) = 1.$

It is immediate that a fuzzy coimplication is an coimplicator analogously defined as a fuzzy implication, replacing ***I*3** and ***I*4** in Definition 1 by

J3: $J(x, 0) = 0$ and

J4: $J(1, y) = 0$, respectively.

There exist many classes of fuzzy (co)implication functions (see, e.g., [15] and [9]). In this paper we consider the class of (S, N)-implications defined in [16] as follow:

$$I_{S,N}(x, y) = S(N(x), y),\ \ \forall x, y \in U, \tag{2.4}$$

such that S is a t-conorm and N is a fuzzy negation. If N is a SFN, then $I_{S,N}$ is called a strong implication or an S-implication.

Additionally, when S_N is the N-dual function of the t-conorm S, the corresponding N-dual functions are **(S,N)-coimplications** given by

$$(I_{S,N})_N(x, y) = S_N(N(x), y),\ \ \forall x, y \in U, \tag{2.5}$$

The dual construction (T, N)-coimplication can be analogously defined.

2.2 Intuitionistic Fuzzy Connectives

These preliminaries consider the IFC concepts in IFL, by applying the strategy to deal with logical connectives as algebraic mappings, meaning that desirable logical and intuitionistic properties are described in terms of algebraic properties of connectives (negation, conjunction, disjunction, implication and coimplication).

According to [1], an Atanassov's fuzzy intuitionistic fuzzy set (IFS) A_I in a non-empty, universe χ, is expressed as

$$A_I = \big\{(x, \mu_A(x), \nu_A(x)) : x \in \chi, \mu_A(x) + \nu_A(x)) \leq 1\big\}.$$

Thus, an intuitionistic fuzzy truth value of an element x in an IFS A_I is related to the ordered pair $(\mu_A(x), \nu_A(x))$. Moreover, an IFS A_I generalizes a fuzzy set $A = \{(x, \mu_A(x))|x \in \chi\}$, since $\nu_A(x)$, which means that the non-membership degree of an element x is less than or equal to the complement of its membership degree $\mu_A(x)$, and therefore $\nu_A(x)$ is not necessarily equal to $1 - \mu_A(x)$.

Additionally, a function $\pi_A : \chi \to U$, called an Atanassov's intuitionistic fuzzy index (IFIx) of an element x in an IFS **A**, is given as

$$\pi_A(x) = N_S(\mu_A(x) + \nu_A(x)) \tag{2.6}$$

Let $\tilde{U} = \{(x_1, x_2) \in U^2 : x_1 \leq N_S(x_2)\}$ be the set of all intuitionistic fuzzy values and $l_{\tilde{U}}, r_{\tilde{U}} : \tilde{U} \to U$ be the projection functions on $\tilde{U}$, which are given by $l_{\tilde{U}}(\tilde{x}) = l_{\tilde{U}}(x_1, x_2) = x_1$ and $r_{\tilde{U}}(\tilde{x}) = r_{\tilde{U}}(x_1, x_2) = x_2$, respectively.

Thus, for all $\tilde{\mathbf{x}} = (\tilde{x}_1, \ldots, \tilde{x}_n) \in \tilde{U}^n$, such that $\tilde{x}_i = (x_{i1}, x_{i2})$ and $x_{i1} \leq N_S(x_{i2})$ when $1 \leq i \leq n$, consider $l_{\tilde{U}^n}, r_{\tilde{U}^n} : \tilde{U}^n \to U^n$ as the projections given by:

$$l_{\tilde{U}^n}(\tilde{\mathbf{x}}) = (l_{\tilde{U}}(\tilde{x}_1), l_{\tilde{U}}(\tilde{x}_2), \ldots, l_{\tilde{U}}(\tilde{x}_n)) = (x_{11}, x_{21}, \ldots x_{n1})$$

$$r_{\tilde{U}^n}(\tilde{\mathbf{x}}) = (r_{\tilde{U}}(\tilde{x}_1), r_{\tilde{U}}(\tilde{x}_2), \ldots r_{\tilde{U}}(\tilde{x}_n)) = (x_{12}, x_{22}, \ldots x_{n2}).$$

Consider also the order relation $\tilde{x} \leq_{\tilde{U}} \tilde{y} \Leftrightarrow x_1 \leq y_1$ and $x_2 \geq y_2$ such that $\tilde{0} = (0, 1) \leq_{\tilde{U}} \tilde{x}$ and $\tilde{1} = (1, 0) \geq_{\tilde{U}} \tilde{x}$, for all $\tilde{x}, \tilde{y} \in \tilde{U}$ [2].

2.2.1 Intuitionistic fuzzy negations

An *Atanassov's intuitionistic fuzzy negation* (IFN shortly) $N_I : \tilde{U} \to \tilde{U}$ satisfies, for all $\tilde{x}, \tilde{y} \in \tilde{U}$, the following properties:

$\mathbf{N}_I$1: $N_I(\tilde{0}) = N_I(0, 1) = \tilde{1}$ and $N_I(\tilde{1}) = N_I(1, 0) = \tilde{0}$;

$\mathbf{N}_I$2: If $\tilde{x} \geq \tilde{y}$ then $N_I(\tilde{x}) \leq N_I(\tilde{y})$.

In addition, N_I is a **strong Atanassov's intuitionistic fuzzy negation** (SIFN) if it also verifies the involutive property:

$\mathbf{N}_I$3: $N_I(N_I(\tilde{x})) = \tilde{x}, \forall \tilde{x} \in \tilde{U}$.

Consider N_I as IFN in $\tilde{U}$ and $\tilde{f} : \tilde{U}^n \to \tilde{U}$. For all $\tilde{\mathbf{x}} = (\tilde{x}_1, \ldots, \tilde{x}_n) \in \tilde{U}^n$, the **$N_I$-dual intuitionistic function of** $\tilde{f}$, denoted by $\tilde{f}_{N_I} : \tilde{U}^n \to \tilde{U}$, is given by:

$$\tilde{f}_{N_I}(\tilde{\mathbf{x}}) = N_I(\tilde{f}(N_I(\tilde{x}_1), \ldots, N_I(\tilde{x}_n))). \tag{2.7}$$

In addition, when $\tilde{N}_I$ is a SIFN, $\tilde{f}$ is a self-dual intuitionistic function.

By [5, Theorem 1] [11, 12], a function $N_I : \tilde{U} \to \tilde{U}$ is a strong intuitionistic fuzzy negation (SIFN) if and only if there exists a (SFN) $N : U \to U$ expressed as:

$$N_I(\tilde{x}) = (N(N_S(x_2)), N_S(N(x_1))), \tag{2.8}$$

Additionally, if $N = N_S$, Eq. (2.8) can be reduced to

$$N_I(\tilde{x}) = (x_2, x_1). \tag{2.9}$$

2.2.2 Intuitionistic fuzzy t-(co)norms

A function $(S_I)T_I : \tilde{U}^2 \to \tilde{U}$ is an Atanassov's intuitionistic fuzzy triangular (co)norm (t-(co)norm shortly), if it is a commutative, associative and increasing function with neutral element $(\tilde{0})$ $\tilde{1}$.

Consider now the t-representability concept proposed in [12, Def. 5], see also some results of [5, Def. 3]. An intuitionistic t-conorm $S_I : \tilde{U}^2 \to \tilde{U}$ and t-norm $T_I : \tilde{U}^2 \to \tilde{U}$ is **t-representable** when both conditions are held:

(i) there exist t-norms $T', T : U^2 \to U$ and t-conorms $S', S : U^2 \to U$ such that, for all $x, y \in U$, the respective equations

$$T'(x, y) \leq N_S(S'(N_S(x), N_S(y))) \text{ and } T(x, y) \leq N_S(S(N_S(x), N_S(y))) \tag{2.10}$$

are verified; and

(ii) for all $\tilde{x} = (x_1, x_2), \tilde{y} = (y_1, y_2) \in \tilde{U}$, each one of such intuitionistic fuzzy connectives is given by the corresponding expressions

$$S_I(\tilde{x}, \tilde{y}) = (S'(x_1, y_1), T'(x_2, y_2)) \text{ and } T_I(\tilde{x}, \tilde{y}) = (T(x_1, y_1), S(x_2, y_2)). \tag{2.11}$$

2.2.3 Intuitionistic fuzzy implications

A binary function $I_I : \widetilde{U}^2 \to \widetilde{U}$ satisfying the conditions:

$\mathbf{I_I 0}$: $I_I(\tilde{0}, \tilde{0}) = I_I(\tilde{0}, \tilde{1}) = I_I(\tilde{1}, \tilde{1}) = \tilde{1}$ and $I_I(\tilde{1}, \tilde{0}) = \tilde{0}$;

is called an Atanassov's intuitionistic fuzzy implicator.

According with [9, Definition 3], an Atanassov's intuitionistic fuzzy implication $I_I : \widetilde{U}^2 \to \widetilde{U}$ is an Atanassov's intuitionistic fuzzy implicator such that, the analogous conditions from $\mathbf{I_I 1}$ to $\mathbf{I_I 4}$ reported in Definition 1 are verified together with the additional property:

$\mathbf{I_I 5}$: If $\tilde{x} = (x_1, x_2)$ such that $\tilde{y} = (y_1, y_2) \in \tilde{U}$, $x_1 + x_2 = 1$ and $y_1 + y_2 = 1$, it holds that $\pi_{I_I((x_1,x_2),(y_1,y_2))} = 0$, with $\pi : \chi \to U$ is the IFIx given by Eq. (2.6).

Thus, recovering Definition 1 of a fuzzy implication in the sense of J. Fodor and M. Roubens' work [16], an Atanassov's intuitionistic fuzzy implication also reproduces fuzzy (co)implications if, for all $\tilde{x} = (x_1, x_2)$, $\tilde{y} = (y_1, y_2) \in \widetilde{U}$ we have $x_1 = N_S(x_2)$ and $y_1 = N_S(y_2)$. According to [2] and [12], another way of defining an operator I_I is to consider boundary conditions in $\mathbf{I}_I0$ and properties $\mathbf{I}_I1$ and $\mathbf{I}_I2$.

Based on [5, Theorem 4] and [11], a function $I_I : \widetilde{U}^2 \to \widetilde{U}$ is a **representable Atanassov's intuitionistic (S, N)-implication** based on a strong negation $N_I : \widetilde{U} \to \widetilde{U}$ if and only if there exist (S, N)-implications $I_a, I_b : U^2 \to U$, such that for all $\tilde{x} = (x_1, x_2)$, $\tilde{y} = (y_1, y_2) \in U$, I_I is expressed as:

$$I_I(\tilde{x}, \tilde{y}) = (I_a(N_S(x_2), y_1),\ N_S(I_b(x_1, N_S(y_2)))). \tag{2.12}$$

Dually, in the same manner, an Atanassov's intuitionistic fuzzy coimplication J_I can be defined. Moreover, a function $J_I : \widetilde{U}^2 \to \widetilde{U}$ is a representable intuitionistic (T, N)-coimplication based on a strong negation N_I iff there exist (T, N)-coimplications $J_a, J_b : U^2 \to U$, such that for all $\tilde{x} = (x_1, x_2)$, $\tilde{y} = (y_1, y_2) \in U$, J_I is expressed as:

$$J_I(\tilde{x}, \tilde{y}) = (J_b(N_S(x_2), y_1),\ N_S(J_a(x_1, N_S(y_2)))). \tag{2.13}$$

When $N = N_S$ and $J_a = I_{aN}$ and $J_b = I_{bN}$, it holds that $J_I = I_{I N_I}$ is a self N_I-dual intuitionistic fuzzy operator.

3 POINTWISE SENSITIVITY OF FUZZY CONNECTIVES

The selection of values using e.g. Likert's scale is an important way to elicit degrees of uncertainty related to based-rule fuzzy systems, frequently expressed by composition performed on fuzzy connectives. Thus, it is desirable that the result of the fuzzy logical operation does not change much if slight changes (or small deviations) are performed in the inputs. This sensitive study leads to the least sensitive or the most robust fuzzy inference rule.

Based on [19] and [25], the study of a δ-sensitivity of n-order function f at point $\mathbf{x}$ on the domain U is considered in the following, in the context of robustness of fuzzy logic, mainly related to the class of (S, N)-implications.

Definition 2. *[19, Def. 1] Let $f : U^n \to U$ be an n-order function, $\delta \in U$ and $\mathbf{x} = (x_1, x_2, \ldots x_n)$, $\mathbf{y} = (y_1, y_2, \ldots y_n) \in U^n$. The **$\delta$-sensitivity of f at point** $\mathbf{x}$, denoted by $\Delta_f(\mathbf{x}, \delta)$, is defined by*

$$\Delta_f(\mathbf{x}, \delta) = \sup\{|f(\mathbf{x}) - f(\mathbf{y})| : \mathbf{y} \in U^n \text{ and } \bigvee(\mathbf{x}, \mathbf{y}) \leq \delta\} \tag{3.1}$$

wherever $\bigvee(\mathbf{x}, \mathbf{y}) = \max\{|x_i - y_i| : i = 1, \ldots, n\}$.

Now, we investigate the δ-sensitivity in FCs, in terms of Definition 2. For that, the binary minimal and maximal fuzzy aggregations were considered: $\wedge, \vee : U^2 \to U$ such that $\min(a, b) = a \wedge b$ and $\max(a, b) = a \vee b$, respectively.

The δ-sensitivity of binary functions at point $\mathbf{x} \in U^2$ are based on the monotonicity property analysis performed in their both arguments. In this paper, such analysis considers the t-(co)norms and fuzzy (co)implications. Thus, three more intuitive results related to the δ-sensitivity of these fuzzy connectives, previously presented in [19], are reported in the sequence, which are also followed by a brief discussion including some exemplification.

Proposition 1. *[19, Theorem 2] Let $f : U \to U$ be a reverse order function, i.e., $x \leq y \Rightarrow f(x) \geq f(y)$, for all $x, y \in U$. The δ-sensitivity of f at point x is given by*

$$\Delta_f(x, \delta) = [f(x) - f((x + \delta) \wedge 1)] \vee [f((x - \delta) \vee 0) - f(x)], \tag{3.2}$$

for $\delta \in U$. In particular, Eq. (3.2) holds for a fuzzy negation function.

Henceforth, in order to provide an easier notation, when $f : U^2 \to U$ and $\mathbf{x} = (x, y) \in U^2$, consider the following notations:

$$f\lfloor\mathbf{x}\rfloor \equiv f((x - \delta) \vee 0, (y - \delta) \vee 0); \qquad f\lfloor\mathbf{x}\rceil \equiv f((x - \delta) \vee 0, (y + \delta) \wedge 1);$$
$$f\lceil\mathbf{x}\rfloor \equiv f((x + \delta) \wedge 1, (y - \delta) \vee 0); \qquad f\lceil\mathbf{x}\rceil \equiv f((x + \delta) \wedge 1, (y + \delta) \wedge 1).$$

Proposition 2. *[19, Theorem 1] Consider $f : U^2 \to U$, $\delta \in U$ and $\mathbf{x} = (x, y) \in U^2$.*

(i) *If f is a monotone function, i.e, $x \leq x'$, $y \leq y' \Rightarrow f(x, y) \leq f(x', y')$ for all $x, y \in U$, then it follows that*

$$\Delta_f(\mathbf{x}, \delta) = (f(\mathbf{x}) - f\lfloor\mathbf{x}\rfloor) \vee (f\lceil\mathbf{x}\rceil - f(\mathbf{x})) \tag{3.3}$$

(ii) *If f verifies both properties, 1-place antitonicity and 2-place isotonicity, then:*

$$\Delta_f(\mathbf{x}, \delta) = (f(\mathbf{x}) - f\lceil\mathbf{x}\rfloor) \vee (f\lfloor\mathbf{x}\rceil - f(\mathbf{x})) \tag{3.4}$$

Based on [19], Proposition 3 formalizes a consequence of Proposition 2:

Proposition 3. *[19, Corollary 1] Let T, S and $I_{S,N}$ be a t-norm, t-conorm and an (S, N)-implication. When $\mathbf{x} \in U^2$ and $\delta \in U$, the next statements are true:*

(i) *the δ-sensitivity of a t-norm T and a t-conorm S at point $\mathbf{x}$, respectively, are both defined by Eq. (3.3);*

(ii) *the δ-sensitivity of an (S, N)-implication $I_{S,N}$ at point $\mathbf{x}$, is defined by Eq. (3.4).*

Remark 1. Let $\delta \in U$. Based on Eq. (3.3), we have the next δ-sensitivity analysis:

(i) when $\mathbf{x} = (0, 1)$, the following is true:

$$\Delta_T((0, 1), \delta) = \delta = \Delta_S((0, 1), \delta); \tag{3.5}$$

(ii) when $\mathbf{x} = (1, 0)$, by the commutativity of a t-(co)norm, we have the same results:

$$\Delta_T((1, 0), \delta) = \delta = \Delta_S((1, 0), \delta) \tag{3.6}$$

(iii) when $\mathbf{x} = (0, 0)$ then the related expressions of a t-(co)norm are given as:

$$\Delta_T((0,0),\delta) = T(\delta,\delta) \quad \text{and} \quad \Delta_S((0,0),\delta) = S(\delta,\delta). \tag{3.7}$$

Additionally, based on Eq. (3.4) and taking $\mathbf{x} = (0, 1)$ and $\mathbf{x} = (1, 0)$, we obtain the corresponding equalities:

$$\Delta_{I_{(S,N)}}((0,1),\delta) = 1 - I_{(S,N)}(\delta, 1-\delta); \tag{3.8}$$

$$\Delta_{I_{(S,N)}}((1,0),\delta) = I_{(S,N)}(1-\delta, \delta). \tag{3.9}$$

3.1 Pointwise sensitivity of N-dual fuzzy connectives

In the preceding section and according with the duality principle stated in Eq. (2.2), we described the definitions as foundations to study the pointwise sensitivity of N-dual FCs as follows.

Proposition 4. *[25, Proposition 6] Let $f : U^2 \to U$ be a second-order function and N be the standard fuzzy negation. For all $\mathbf{x} = (x, y) \in U^n$, the following equalities hold:*

(i) $f_N\lfloor\mathbf{x}\rfloor = N(f\lceil N(\mathbf{x})\rceil)$; (ii) $f_N\lfloor\mathbf{x}\rceil = N(f\lceil N(\mathbf{x})\rfloor)$;

(iii) $f_N\lceil\mathbf{x}\rfloor = N(f\lfloor N(\mathbf{x})\rceil)$; (iv) $f_N\lceil\mathbf{x}\rceil = N(f\lfloor N(\mathbf{x})\rfloor)$.

Taken at a strong fuzzy negation N, Proposition 5 states that the sensitivity of a n-order function f at a point $\mathbf{x}$ is equal to the sensitivity of its dual function f_N taking the complement of $\mathbf{x}$.

Proposition 5. *[25, Theorem 1] Consider $f : U^2 \to U$, $\delta \in U$ and $\mathbf{x} = (x, y) \in U^2$. Let $\Delta_f(\mathbf{x}, \delta)$ be the sensitivity of f at point $\mathbf{x}$. If N is the standard fuzzy negation ($N = N_S$ in Eq. (2.1)) and f_N is the N-dual function of f then **the sensitivity of** f_N **at point** $\mathbf{x}$ is given by*

$$\Delta_{f_N}(\mathbf{x},\delta) = \Delta_f(N(\mathbf{x}),\delta). \tag{3.10}$$

Proposition 6. *[25, Proposition 7] Let N be the standard fuzzy negation, f_N be N-dual function related to $f : U^2 \to U$, $\delta \in U$ and $\mathbf{x} = (x, y) \in U^2$. The sensitivity of f_N at point $\mathbf{x}$ is given by the following cases:*

(i) *if f is increasing w.r.t. its variables then we have that:*

$$\Delta_{f_N}(\mathbf{x},\delta) = (f_N\lceil\mathbf{x}\rceil - f_N(\mathbf{x})) \vee (f_N\lfloor\mathbf{x}\rfloor - f_N(\mathbf{x})); \tag{3.11}$$

(ii) *if f is decreasing w.r.t. its first variable and increasing with its second variable then we have:*

$$\Delta_{f_N}(\mathbf{x},\delta) = (f_N\lfloor\mathbf{x}\rceil - f_N(\mathbf{x})) \vee (f_N\lceil\mathbf{x}\rfloor - f_N(\mathbf{x})). \tag{3.12}$$

Proposition 7. *[25, Proposition 8] Let $(T)_N$, $(S)_N$, $(I_{S,N})_N$ be N-dual functions related to a t-norm T, a t-conorm S and an implication $I_{S,N}$, respectively. If $\mathbf{x} \in U^2$ and $\delta \in U$, the statements as follow hold:*

(i) $\Delta_{(T)_N}(\mathbf{x}, \delta)$ *and* $\Delta_{(S)_N}(\mathbf{x}, \delta)$ *are both defined by Eq. (3.11);*

(ii) $\Delta_{I_{(S,N)}{}_N}(\mathbf{x}, \delta)$ *is defined by Eq. (3.12).*

Remark 2. Consider $\delta \in U$ and the pair $(I_{(S,N)}, I_{(S,N)_N})$ of mutual N-dual functions. By Eqs. (3.8) and (3.9) stated in Remark 1, it follows the expressions:

$$\Delta_{I_{(S,N)}{}_N}((1,0),\delta) = I_{(S,N)_N}(1-\delta,\delta) = 1 - I_{(S,N)}(\delta, 1-\delta) = 1 - \Delta_{I_{(S,N)}}((0,1),\delta);$$
$$\Delta_{I_{(S,N)}{}_N}((0,1),\delta) = 1 - I_{(S,N)_N}(\delta, 1-\delta) = I_{(S,N)}(1-\delta,\delta) = \Delta_{I_{(S,N)}}((1,0),\delta);$$

which are examples of Eq. (3.12) when $\mathbf{x} = (1,0)$ and $\mathbf{x} = (0,1)$, respectively.

4 ROBUSTNESS OF INTUITIONISTIC FUZZY CONNECTIVES

The sensitivity of fuzzy connectives contributes to measure the robustness of fuzzy reasoning directly linked to the selection of implication operators. When a fuzzy connective is modelled by a continuous function f_I, one can consider the modulus of continuity of f_I, by using a modulus of continuity of f_I to any n-order intuitionistic fuzzy connective.

In order to provide a formal definition of robustness which can be applied to n-order Atanassov's intuitionistic fuzzy operators (as averages and medians aggregation functions) we introduce the definition of the δ-sensitivity of $f : \tilde{U}^n \to \tilde{U}$ at point $\tilde{\mathbf{x}} = (\tilde{x}_1, \tilde{x}_2, \ldots, \tilde{x}_n) \in \tilde{U}^n$.

Thus, δ-sensitivity of an intuitionistic operator f at point $\tilde{\mathbf{x}} \in \tilde{U}^n$ is defined in terms of its left-projection $(l_{\tilde{U}^n}(\tilde{\mathbf{x}}))$ and right-projection $(r_{\tilde{U}^n}(\tilde{\mathbf{x}}))$, which are related to the δ-sensitive of the membership and non-membership degrees of an element $x \in \chi$ associated with the IFS $f(\tilde{U}^n)$.

Definition 3. *Let* $f_I : \tilde{U}^n \to \tilde{U}$ *be an n-order function,* $\delta = (\delta_1, \delta_2) \in U^2$ *and* $\tilde{\mathbf{x}} = (\tilde{x}_1, \tilde{x}_2, \ldots, \tilde{x}_n) \in \tilde{U}^n$. *The* ***δ-sensitivity of*** f_I ***at point*** $\tilde{\mathbf{x}}$, *denoted by* $\Delta_{f_I}(\tilde{\mathbf{x}}, \delta)$, *is defined by*

$$\Delta_{f_I}(\tilde{\mathbf{x}}, \delta) = \sup\Big\{ |f_I(\tilde{\mathbf{x}}) - f_I(\tilde{\mathbf{y}})| : \tilde{\mathbf{y}} \in \tilde{U}^n \text{ and } \bigvee(l_{\tilde{U}^n}(\tilde{\mathbf{x}}), l_{\tilde{U}^n}(\tilde{\mathbf{y}})) \leq \delta_1 \text{ and } \bigvee(r_{\tilde{U}^n}(\tilde{\mathbf{x}}), r_{\tilde{U}^n}(\tilde{\mathbf{y}})) \leq \delta_2 \Big\}. \tag{4.1}$$

wherever $\bigvee(\mathbf{x}, \mathbf{y}) = \max\{|x_i - y_i| : i = 1, \ldots, n\}$.

The next proposition states that pointwise sensitivity is preserved by the projection functions applied to intuitionistic fuzzy negation that is t-representable in the same sense of [5, 12] and [11].

Proposition 8. *Let* $N_I : \tilde{U} \to \tilde{U}$ *be a representable intuitionistic negation as defined by Eq. (2.8). When* $\delta = (\delta_1, \delta_2) \in U^2$ *and* $\tilde{\mathbf{x}} \in \tilde{U}^2$, *the* ***δ-sensitivity of*** N_I ***at point*** $\tilde{\mathbf{x}}$, *is defined by*

$$\Delta_{N_I}(\tilde{\mathbf{x}}, \delta) = (\Delta_N(l_{\tilde{U}^2}(\tilde{\mathbf{x}}), \delta_1), \Delta_N(r_{\tilde{U}^2}(\tilde{\mathbf{x}}), \delta_2)). \tag{4.2}$$

Proof. Straightfoward from Definition 3 and Proposition 1. □

In the following, the study of a δ-sensitivity of an Atanassov's intuitionistic t-(co)norm and an intuitionistic fuzzy implication at point $\tilde{\mathbf{x}} = (\mathbf{x}_1, \mathbf{x}_2) = ((x_{11}, x_{12}), (x_{11}, x_{12}))$ on the domain $\tilde{U}^2$ extends the work introduced in [19] related to the class of such binary intuitionistic fuzzy connectives which are representable by their corresponding fuzzy connectives [5, 11] and [12].

Proposition 9. *Let $(S_I)T_I : \tilde{U}^2 \to \tilde{U}$ be a representable Atanassov's intuitionistic fuzzy t-(co)norm as defined by (Eq. (2.11b)) Eq. (2.11a). When $\delta = (\delta_1, \delta_2) \in U^2$ and $\tilde{\mathbf{x}} \in \tilde{U}^2$, the δ* ***sensitivity of T_I at point*** *$\tilde{\mathbf{x}}$, denoted by $\Delta_{T_I}(\tilde{\mathbf{x}}, \delta)$, is given by*

$$\Delta_{T_I}(\tilde{\mathbf{x}}, \delta) = (\Delta_T(l_{\tilde{U}^2}(\tilde{\mathbf{x}}), \delta_1), \Delta_S(r_{\tilde{U}^2}(\tilde{\mathbf{x}}), \delta_2)). \tag{4.3}$$

Analogously, the δ ***sensitivity of S_I at point*** *$\tilde{\mathbf{x}}$ is given by*

$$(\Delta_{S_I}(\tilde{\mathbf{x}}, \delta) = (\Delta_S(l_{\tilde{U}^2}(\tilde{\mathbf{x}}), \delta_1), \Delta_T(r_{\tilde{U}^2}(\tilde{\mathbf{x}}), \delta_2))); \tag{4.4}$$

Proof. For all $\tilde{\mathbf{x}}, \tilde{\mathbf{y}} \in \tilde{U}^2$ given as $\tilde{\mathbf{x}} = (\tilde{x}_1, \tilde{x}_2)$, such that $\tilde{x}_1 = (x_{11}, x_{12})$, $x_{11} \le N_S(x_{12})$ and $\tilde{x}_2 = (x_{21}, x_{22})$, $x_{21} \le N_S(x_{22})$; $\tilde{\mathbf{y}} = (\tilde{y}_1, \tilde{y}_2)$, such that $\tilde{y}_1 = (y_{11}, y_{12})$, $y_{11} \le N_S(y_{12})$ and $\tilde{y}_2 = (y_{21}, y_{22})$, $y_{21} \le N_S(y_{22})$. It holds that:

$$\begin{aligned}
&\Delta_{T_I}(\tilde{\mathbf{x}}, \delta) = \\
&= \sup\Big\{|T_I(\tilde{\mathbf{x}}) - T_I(\tilde{\mathbf{y}})| : \tilde{\mathbf{y}} \in \tilde{U}^n \text{ and} \\
&\quad \bigvee(l_{\tilde{U}^n}(\tilde{\mathbf{x}}), l_{\tilde{U}^n}(\tilde{\mathbf{y}})) \le \delta_1 \text{ and } \bigvee(r_{\tilde{U}^n}(\tilde{\mathbf{x}}), r_{\tilde{U}^n}(\tilde{\mathbf{y}})) \le \delta_2\Big\} \text{ by Eq. (4.1)} \\
&= \sup\Big\{|T_I((x_{11}, x_{12}), (x_{21}, x_{22})) - T_I((y_{11}, y_{12}), (y_{21}, y_{22}))| : \tilde{\mathbf{y}} \in \tilde{U}^n \text{ and} \\
&\quad \bigvee(l_{\tilde{U}^n}(\tilde{\mathbf{x}}), l_{\tilde{U}^n}(\tilde{\mathbf{y}})) \le \delta_1 \text{ and } \bigvee(r_{\tilde{U}^n}(\tilde{\mathbf{x}}), r_{\tilde{U}^n}(\tilde{\mathbf{y}})) \le \delta_2\Big\} \\
&= \sup\Big\{|(T(x_{11}, x_{21}), S(x_{12}, x_{22})) - (T(y_{11}, y_{21}), S(y_{12}, y_{22}))| : \tilde{\mathbf{y}} \in \tilde{U}^n \text{ and} \\
&\quad \bigvee(l_{\tilde{U}^n}(\tilde{\mathbf{x}}), l_{\tilde{U}^n}(\tilde{\mathbf{y}})) \le \delta_1 \text{ and } \bigvee(r_{\tilde{U}^n}(\tilde{\mathbf{x}}), r_{\tilde{U}^n}(\tilde{\mathbf{y}})) \le \delta_2\Big\} \text{ by Eq. (2.11 b)} \\
&= \Big(\sup\Big\{|T(x_{11}, x_{21}) - T(y_{11}, y_{21})| : \tilde{\mathbf{y}} \in \tilde{U}^n \text{ and } \bigvee(l_{\tilde{U}^n}(\tilde{\mathbf{x}}), l_{\tilde{U}^n}(\tilde{\mathbf{y}})) \le \delta_1\Big\}, \\
&\quad \sup\Big\{|S(y_{12}, y_{22}) - S(x_{12}, x_{22})| : \tilde{\mathbf{y}} \in \tilde{U}^n \text{ and } \bigvee(r_{\tilde{U}^n}(\tilde{\mathbf{x}}), r_{\tilde{U}^n}(\tilde{\mathbf{y}})) \le \delta_2\Big\}\Big) \\
&= \Delta_T(l_{\tilde{U}}(\tilde{\mathbf{x}}, \delta_1), \Delta_S(r_{\tilde{U}}(\tilde{\mathbf{x}}), \delta_2)) \text{ by Eq. (3.1)}
\end{aligned}$$

Therefore, $l_{\tilde{U}^2}(\Delta_{T_I}(\tilde{\mathbf{x}}, \delta)) = \Delta_T(l_{\tilde{U}^2}(\tilde{\mathbf{x}}), \delta_1)$ and $r_{\tilde{U}^2}(\Delta_{T_I}(\tilde{\mathbf{x}}, \delta)) = \Delta_S(r_{\tilde{U}^2}(\tilde{\mathbf{x}}), \delta_2)$. Analogously, it can be proved that for δ-sensitivity of S_I at point $\tilde{\mathbf{x}}$ which means, $l_{\tilde{U}^2}(\Delta_{S_I}(\tilde{\mathbf{x}}, \delta)) = \Delta_S(l_{\tilde{U}^2}(\tilde{\mathbf{x}}), \delta_1)$ and $r_{\tilde{U}^2}(\Delta_{S_I}(\tilde{\mathbf{x}}, \delta)) = \Delta_T(r_{\tilde{U}^2}(\tilde{\mathbf{x}}), \delta_2)$. □

Remark 3. According with Eqs. (4.4) and (4.3) in Proposition 10, we obtain the expressions of the δ-sensitivity of an intuitionistic fuzzy t-(co)norm as follows:

(i) when $\tilde{\mathbf{x}} = ((0,0),(0,0))$ and $\tilde{\delta} = (\delta_1, \delta_2)$, it holds that

$$\Delta_{T_I}(\tilde{\mathbf{x}}, \delta) = (T(\delta_1, \delta_1), S(\delta_2, \delta_2)); \ \ \Delta_{S_I}(\tilde{\mathbf{x}}, \delta) = (S(\delta_1, \delta_1), T(\delta_2, \delta_2)).$$

(ii) when $\tilde{\mathbf{x}} = (\tilde{1}, \tilde{1})$ or $\tilde{\mathbf{x}} = (\tilde{0}, \tilde{0})$ and $\tilde{\delta} = (\delta_1, \delta_2)$, $\Delta_{T_I}(\tilde{\mathbf{x}}, \delta) = \delta = \Delta_{S_I}(\tilde{\mathbf{x}}, \delta)$.

Now, we study the robustness of an (S, N)-implication I_I at point $\tilde{\mathbf{x}} \in \tilde{U}^2$.

Proposition 10. *Let $(J_I) I_I : \tilde{U}^2 \to \tilde{U}$ be a t-representable Atanassov's intuitionistic ((T, N)-implication) (S, N)-implication as defined by (Eq. (2.13)) Eq. (2.12). When $\tilde{\delta} = (\delta_1, \delta_2) \in U^2$ and $\tilde{\mathbf{x}} \in \tilde{U}^2$, the δ-**sensitivity of** I_I **at point** $\tilde{\mathbf{x}}$ is defined by*

$$\Delta_{I_I}(\tilde{\mathbf{x}}, \tilde{\delta}) = (\Delta_{I_a}(l_{\tilde{U}^2}(\tilde{\mathbf{x}}, \delta_1), \Delta_{I_b}(r_{\tilde{U}^2}(\tilde{\mathbf{x}})).\delta_2)) \tag{4.5}$$

*Analogously, the δ-**sensitivity of** J_I **at point** $\tilde{\mathbf{x}}$ is defined by*

$$(\Delta_{J_I}(\tilde{\mathbf{x}}, \tilde{\delta}) = (\Delta_{J_b}(l_{\tilde{U}^2}(\tilde{\mathbf{x}}), \delta_1), \Delta_{J_a}(r_{\tilde{U}^2}(\tilde{\mathbf{x}})), \delta_2)) \tag{4.6}$$

Proof. Let I_I be a representable (S, N)-implication obtained by the standard fuzzy negation N_S and a fuzzy (S, N)-implications I_a, I_b, as defined by Eq. (2.12), then:

$$\begin{aligned}
&\Delta_{I_I}(\tilde{\mathbf{x}}, \tilde{\delta}) = \\
&= \sup\Big\{|I_I(\tilde{\mathbf{x}}) - I_I(\tilde{\mathbf{y}})| : \tilde{\mathbf{y}} \in \tilde{U}^2 \text{ and} \\
&\quad \bigvee(l_{\tilde{U}^2}(\tilde{\mathbf{x}}), l_{\tilde{U}^2}(\tilde{\mathbf{y}})) \leq \delta_1 \text{ and } \bigvee(r_{\tilde{U}^2}(\tilde{\mathbf{x}}), r_{\tilde{U}^2}(\tilde{\mathbf{y}})) \leq \delta_2\Big\} \text{ by Eq. (4.1)} \\
&= \sup\Big\{|I_I((x_{11}, x_{12}), (x_{21}, x_{22})) - I_I((y_{11}, y_{12}), (y_{21}, y_{22}))| : \tilde{\mathbf{y}} \in \tilde{U}^2 \text{ and} \\
&\quad \bigvee(l_{\tilde{U}^2}(\tilde{\mathbf{x}}), l_{\tilde{U}^2}(\tilde{\mathbf{y}})) \leq \delta_1 \text{ and } \bigvee(r_{\tilde{U}^2}(\tilde{\mathbf{x}}), r_{\tilde{U}^2}(\tilde{\mathbf{y}})) \leq \delta_2\Big\} \\
&= \sup\Big\{|(I_a(N_S(x_{12}), x_{21}), N_S(I_b(x_{11}, N_S(x_{22}))) - (I_a(N_S(y_{12}), y_{21}), N_S(I_b(y_{11}, N_S(y_{22})))| : \\
&\quad \tilde{\mathbf{y}} \in \tilde{U}^2 \text{ and } \bigvee(l_{\tilde{U}^n}(\tilde{\mathbf{x}}), l_{\tilde{U}^2}(\tilde{\mathbf{y}})) \leq \delta_1 \text{ and } \bigvee(r_{\tilde{U}^2}(\tilde{\mathbf{x}}), r_{\tilde{U}^2}(\tilde{\mathbf{y}})) \leq \delta_2\Big\} \text{ by Eq. (2.12)} \\
&= \Big(\sup\Big\{|I_a(N_S(x_{12}), x_{21}) - I_a(N_S(y_{12}), y_{21})| : \tilde{\mathbf{y}} \in \tilde{U}^2 \text{ and } \bigvee(l_{\tilde{U}^2}(\tilde{\mathbf{x}}), l_{\tilde{U}^2}(\tilde{\mathbf{y}})) \leq \delta_1\Big\}, \\
&\quad \sup\Big\{|N_S(I_b(y_{11}, N_S(y_{22}))) - N_S(I_b(x_{11}, N_S(x_{22})))| : \tilde{\mathbf{y}} \in \tilde{U}^2 \text{ and } \bigvee(r_{\tilde{U}^2}(\tilde{\mathbf{x}}), r_{\tilde{U}^2}(\tilde{\mathbf{y}})) \leq \delta_2\Big\}\Big) \\
&= \Big(\sup\Big\{|I_a(N_S(x_{12}), x_{21}) - I_a(N_S(y_{12}), y_{21})| : \tilde{\mathbf{y}} \in \tilde{U}^2 \text{ and } \bigvee(l_{\tilde{U}^2}(\tilde{\mathbf{x}}), l_{\tilde{U}^2}(\tilde{\mathbf{y}})) \leq \delta_1\Big\}, \\
&\quad \sup\Big\{|I_b(x_{11}, N_S(x_{22})) - I_b(y_{11}, N_S(y_{22}))| : \tilde{\mathbf{y}} \in \tilde{U}^2 \text{ and } \bigvee(r_{\tilde{U}^2}(\tilde{\mathbf{x}}), r_{\tilde{U}^2}(\tilde{\mathbf{y}})) \leq \delta_2\Big\}\Big) \\
&= \big(\Delta_{I_a}(l_{\tilde{U}}(\tilde{\mathbf{x}}), \delta_1), \Delta_{I_b}(r_{\tilde{U}}(\tilde{\mathbf{x}}), \delta_2)\big) \text{ by Eq. (3.1).}
\end{aligned}$$

Therefore, for all $\tilde{\mathbf{x}}, \tilde{\mathbf{y}} \in \tilde{U}^2$, it follows that

$$l_{\tilde{U}^2}(\Delta_{I_I}(\tilde{\mathbf{x}}, \tilde{\delta})) = \Delta_{I_a}(l_{\tilde{U}}(\tilde{\mathbf{x}}), \delta_1); \ \ r_{\tilde{U}^2}(\Delta_{I_I}(\tilde{\mathbf{x}}, \delta)) = \Delta_{I_b}(l_{\tilde{U}}(\tilde{\mathbf{x}})), \delta_2).$$

In analogous manner, Eq. (4.6) can be proved. □

Remark 4. Based on results in Remark 2, we are able to analyse the δ-sensitivity of an Atanassov's intuitionistic fuzzy ((T, N)-coimplication) (S, N)-implication (J_I) I_I as follows:

(i) when $\tilde{\mathbf{x}} = (\tilde{0}, \tilde{1})$ and $\tilde{\delta} = (\delta_1, \delta_2)$ then

$$\Delta_{I_I}((\tilde{0}, \tilde{1}), \tilde{\delta}) = \big(\Delta_{I_a}((1,1), \delta_1), \Delta_{I_b}(0,0), \delta_2\big) = \big(1 - I_b(0, \delta_1), I_a(1-\delta_2, 1)\big);$$
$$\Delta_{J_I}((\tilde{0}, \tilde{1}), \tilde{\delta}) = \big(J_b(0, \delta_1), J_a(1-\delta_2, \delta_2)\big).$$

(ii) when $\tilde{\mathbf{x}} = (\tilde{1}, \tilde{0})$ and $\tilde{\delta} = (\delta_1, \delta_2)$ then

$$\Delta_{I_I}((\tilde{1}, \tilde{0}), \tilde{\delta}) = \big(\Delta_{I_a}(l_{\tilde{U}}(\tilde{1}), \delta_1), \Delta_{I_b}(r_{\tilde{U}}(\tilde{0}), \delta_2)\big) = \big(I_a(1-\delta_1, \delta_1), 1 - I_b(\delta_2, 1-\delta_2)\big);$$
$$\Delta_{J_I}((\tilde{1}, \tilde{0}), \tilde{\delta}) = \big(J_a(1-\delta_1, \delta_1), 1 - J_b(\delta_2, 1-\delta_2)\big).$$

4.1 Robustness of N-dual intuitionistic fuzzy connectives

Now, consider $\delta = (\delta_1, \delta_2) \in U^2$, $\tilde{\mathbf{x}} = (\tilde{x}_1, \tilde{x}_2)$, $\tilde{\mathbf{y}} = (\tilde{y}_1, \tilde{y}_2) \in \tilde{U}^n$ in the following.

Proposition 11. *Let $f_I : \tilde{U}^n \to \tilde{U}$ be an n-order function and $\Delta_{f_I}(\tilde{\mathbf{x}}, \delta)$ be the sensitivity of f_I at point $\tilde{\mathbf{x}}$. When $N_I = N_{I\,S}$ (as stated in Eq. (2.9)) and $f_{I\,N_I}$ the N_I-dual function of f_I,* ***the δ-sensitivity of*** *$f_{I\,N_I}$* ***at point*** *$\tilde{\mathbf{x}}$ is given by*

$$\Delta_{f_{I\,N_I}}(\tilde{\mathbf{x}}, \delta) = \Delta_{f_I}(N_I(\tilde{\mathbf{x}}), \delta). \tag{4.7}$$

Proof. Straightforward from Definition 3 and duality principle. □

Proposition 12. *Let $(S_I)T_I : \tilde{U}^n \to \tilde{U}$ be a representable Atanassov's intuitionistic fuzzy t-(co)norm given as Eqs. (2.11a) and (2.11b), and $(\Delta_S(\mathbf{x}, \delta))$ $\Delta_T(\mathbf{x}, \delta)$ be the δ-sensitivity of a t-(co)norm (S) T at point $\mathbf{x}$. When $N = N_S$ and (S_N) T_N is the N-dual function of (S) T, the* ***δ-sensitivity of*** *$T_{I\,N_I}$* ***at point*** *$\tilde{\mathbf{x}}$ is given by*

$$\Delta_{T_{I\,N_S}}(\tilde{\mathbf{x}}, \delta) = \left(\Delta_T(N_I(r_{\tilde{U}^2}(\tilde{\mathbf{x}})), \delta_1), \Delta_S(N_I(l_{\tilde{U}^2}(\tilde{\mathbf{x}})), \delta_2)\right). \tag{4.8}$$

Analogously, the ***δ-sensitivity of*** *$S_{I\,N_I}$* ***at point*** *$\tilde{\mathbf{x}}$ is given by*

$$\Delta_{S_{I\,N_S}}(\tilde{\mathbf{x}}, \delta) = \left(\Delta_S(N_I(r_{\tilde{U}^2}(\tilde{\mathbf{x}})), \delta_1), \Delta_T(N_I(l_{\tilde{U}^2}(\tilde{\mathbf{x}})), \delta_2)\right) \tag{4.9}$$

Proof. Straightforward from Proposition 11. □

Remark 5. Based on results in Propositions 11 and 12 , we are able to analyse the δ-sensitivity of an intuitionistic fuzzy t-(co)norm (S_I) T_I as follows:

(i) when $\tilde{\mathbf{x}} = (\tilde{0}, \tilde{1})$, by Eqs. (3.5) and (3.6) we have that:

$$\Delta_{S_{I\,N_S}}(\tilde{\mathbf{x}}, \delta) = \big(\Delta_S(\tilde{0}, \delta_1), \Delta_T(\tilde{1}, \delta_2)\big) = \delta = \big(\Delta_T(\tilde{1}, \delta_1), \Delta_S(\tilde{0}, \delta_2)\big) = \Delta_{S_I}(N_I(\tilde{\mathbf{x}}), \delta);$$
$$\Delta_{T_{I\,N_S}}(\tilde{\mathbf{x}}, \delta) = \big(\Delta_T(\tilde{0}, \delta_1), \Delta_S(\tilde{1}, \delta_2)\big) = \delta = \big(\Delta_S(\tilde{1}, \delta_1), \Delta_T(\tilde{0}, \delta_2)\big) = \Delta_{T_I}(N_I(\tilde{\mathbf{x}}), \delta).$$

(ii) when $\tilde{\mathbf{x}} = ((0,0),(0,0))$ we have that:

$$\Delta_{S_{I\,N_I}}(\tilde{\mathbf{x}},\delta) = \big(\Delta_T((0,0),\delta_1), \Delta_S((0,0),\delta_2)\big) = \big(S(\delta_1,\delta_1), T(\delta_2,\delta_2)\big) = \Delta_{S_I}(N_I(\tilde{\mathbf{x}}),\delta);$$
$$\Delta_{T_{I\,N_I}}(\tilde{\mathbf{x}},\delta) = \big(\Delta_S((0,0),\delta_1), \Delta_T((0,0),\delta_2)\big) = \big(T(\delta_1,\delta_1), S(\delta_2,\delta_2)\big) = \Delta_{T_I}(N_I(\tilde{\mathbf{x}}),\delta).$$

Proposition 13. *Let $(J_I) I_I : \tilde{U}^n \to \tilde{U}$ be a representable Atanassov's intuitionistic fuzzy (co)implication and $(\Delta_J(\mathbf{x},\delta))$ $\Delta_I(\mathbf{x},\delta)$ be the δ-sensitivity of a (co)implication (J) I at point $\mathbf{x}$. According with Proposition 11, when N is the standard fuzzy negation ($N = N_S$ in Eq. (2.1)) and (J_N) I_N is the N-dual function of (J) I, the δ-**sensitivity of** $I_{I\,N_I}$ **at point** $\tilde{\mathbf{x}}$ is given by*

$$\Delta_{I_{I\,N_I}}(\tilde{\mathbf{x}},\delta) = \big(\Delta_I(N_S(r_{\tilde{U}}(\tilde{\mathbf{x}})),\delta_1), \Delta_J(N_S(l_{\tilde{U}}(\tilde{\mathbf{x}})),\delta_2)\big). \tag{4.10}$$

*Analogously, the δ-**sensitivity of** $J_{I\,N_I}$ **at point** $\tilde{\mathbf{x}}$ is given by*

$$\Delta_{J_{I\,N_I}}(\tilde{\mathbf{x}},\delta) = \big(\Delta_J(N_S(r_{\tilde{U}}(\tilde{\mathbf{x}})),\delta_1), \Delta_I(N_S(l_{\tilde{U}}(\tilde{\mathbf{x}})),\delta_2)\big). \tag{4.11}$$

Proof. Straightforward from Proposition 12. □

Remark 6. Based on results in Propositions 11 and 12, we are able to analyse the δ-sensitivity of an Atanassov's intuitionistic fuzzy t-(co)norm (S_I) T_I as follows:

(i) when $\tilde{\mathbf{x}} = (\tilde{0},\tilde{1})$, according with Remark 4(i) it follows the expressions:

$$\Delta_{I_{I\,N_I}}((\tilde{0},\tilde{1}),\tilde{\delta}) = (\Delta_{I_a}(\tilde{1},\delta_1), \Delta_{I_b}(\tilde{0},\delta_2));$$
$$\Delta_{J_{I\,N_I}}((\tilde{0},\tilde{1}),\tilde{\delta}) = (\Delta_{J_a}(\tilde{1},\delta_1), \Delta_{J_b}(\tilde{0},\delta_2)).$$

(ii) when $\tilde{\mathbf{x}} = (\tilde{1},\tilde{0})$ then by Remark 4(ii) it follows other dual expressions:

$$\Delta_{I_{I\,N_I}}((\tilde{1},\tilde{0}),\tilde{\delta}) = (\Delta_{I_a}(\tilde{0},\delta_1), \Delta_{I_b}(\tilde{1},\delta_2));$$
$$\Delta_{J_{I\,N_I}}((\tilde{1},\tilde{0}),\tilde{\delta}) = (\Delta_{J_a}(\tilde{0},\delta_1), \Delta_{J_b}(\tilde{1},\delta_2)).$$

5 CONCLUSION

The main contribution of this paper is concerned with the study of robustness on Atanassov's intuitionistic fuzzy operators mainly used in fuzzy reasoning based on IFL. Taking the class of strong fuzzy negation (standard negation), the paper formally states that the sensitivity of an n-order Atanassov's intuitionistic fuzzy connective at a point $\mathbf{x} \in U^n$ preserves its projections related to the sensitivity of its fuzzy approach at the same point. The work of estimating its sensitivity to small changes is related to reducing sensitivity in the corresponding fuzzy connectives.

Our current investigation clearly aims to contemplate two approaches: (i) the sensitivity of fuzzy inference dependent on intuitionistic fuzzy rules based on intuitionistic fuzzy connectives; and (ii) the extension of the robustness studies to other main classes of (co)implications: R-(co)implications and QL-(co)implications.

RESUMO. A ánalise da robustez de conectivos fuzzy intuicionista consiste na principal contribuição deste trabalho. A partir da avaliação da sensibilidade de funções n-arias na classe dos conjuntos fuzzy intuicionistas, como proposto por Atanassov, os principais resultados são aplicados na correspondente extensão intuicionista das (S, N)-implicações fuzzy. O trabalho mostra que a robustez preserva as funções de projeções nesta classe de implicações fuzzy.

Palavras-chave: robustez, δ-sensibilidade, perturbação, lógica fuzzy, lógica fuzzy intuicionística, conectivos fuzzy intuicionistas.

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