Tendências em Matemática Aplicada e Computacional, **1**, No. 2 (2000), 373-387.
Sociedade Brasileira de Matemática Aplicada e Computacional.

Analysis of Product Integration Methods for a Class of Singular Volterra Integral Equations

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Abstract. The construction and analysis of high order numerical methods for Volterra integral equations with a certain weakly singular kernel have been investigated in [6], under the assumption that the solution is sufficiently smooth. In the present work we give a more detailed convergence analysis and show how the continuity requirements can be relaxed, in particular by employing the techniques developed in [13]. Several numerical results are presented.

1. Introduction

We consider the Volterra integral equation of the second kind

$$F(t) + \int_0^t P(t,s)F(s)ds = H(t), \quad t \in [0,T], \tag{1.1}$$

where

$$P(t,s) := \frac{1}{\sqrt{\pi}} \frac{1}{\sqrt{\ln(t/s)}} \frac{1}{t} \tag{1.2}$$

and $H(t)$ is a given function. The above equation arises in some heat conduction problems with mixed-type boundary conditions [1]. As an illustration, consider

$$\frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2}, \quad 0 \le x \le l, \tag{1.3}$$

with the conditions

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$$u(x,-\infty)=0, \tag{1.4}$$

$$\frac{\partial u}{\partial x}(0,t)-u(0,t)=\phi_1(t), \tag{1.5}$$

$$-\frac{\partial u}{\partial x}(l,t)-u(l,t)=\phi_2(t). \tag{1.6}$$

The solution $u(x,t)$ can be expressed in terms of single layer potentials (see e.g. [1], [18]) as follows

$$u(x,t)=\frac{a}{2\sqrt{\pi}}\int_{-\infty}^{t}(t-\tau)^{-1/2}\left[\rho_1(\tau)\ e^{\dfrac{-x^2}{4a^2(t-\tau)}}+\rho_2(\tau)\ e^{\dfrac{-(x-l_1)^2}{4a^2(t-\tau)}}\right]d\tau. \tag{1.7}$$

Above $\rho_1(\tau)$, $\rho_2(\tau)$ are such that $u(x,t)$ satisfies conditions (1.4)-(1.6). By imposing those conditions, the following system of two integral equations is obtained

$$\frac{a}{\sqrt{\pi}}\int_0^u\left[\frac{l}{2a^2\sqrt{\ln^3(u/x)}}-\frac{1}{\sqrt{\ln(u/x)}}\right]\frac{1}{x}\ e^{\dfrac{-l^2}{4a^2\ln(u/x)}}\ \psi_2\left(\frac{1}{x}\right)\ dx$$
$$-\frac{a}{\sqrt{\pi}}\int_0^u\frac{1}{\sqrt{\ln(u/x)}}\ \frac{1}{x}\ \psi_1\left(\frac{1}{x}\right)\ dx-\psi_1\left(\frac{1}{u}\right)=H_1\left(\frac{1}{u}\right), \tag{1.8}$$

$$\frac{a}{\sqrt{\pi}}\int_0^u\left[\frac{l}{2a^2\sqrt{\ln^3(u/x)}}-\frac{1}{\sqrt{\ln(u/x)}}\right]\frac{1}{x}\ e^{\dfrac{-l^2}{4a^2\ln(u/x)}}\ \psi_1\left(\frac{1}{x}\right)\ dx$$
$$-\frac{a}{\sqrt{\pi}}\int_0^u\frac{1}{\sqrt{\ln(u/x)}}\ \frac{1}{x}\ \psi_2\left(\frac{1}{x}\right)\ dx-\psi_2\left(\frac{1}{u}\right)=H_2\left(\frac{1}{u}\right), \tag{1.9}$$

where $\psi_k(s):=\rho_k(-\ln s)$, $H_k(s):=2\phi_k(-\ln s)$. If l is large compared to a, then we may consider the system

$$-\frac{a}{\sqrt{\pi}}\int_0^u\frac{1}{\sqrt{\ln(u/x)}}\ \frac{1}{x}\ \psi_1\left(\frac{1}{x}\right)\ dx-\psi_1\left(\frac{1}{u}\right)=H_1\left(\frac{1}{u}\right), \tag{1.10}$$

$$-\frac{a}{\sqrt{\pi}}\int_0^u\frac{1}{\sqrt{\ln(u/x)}}\ \frac{1}{x}\ \psi_2\left(\frac{1}{x}\right)\ dx-\psi_2\left(\frac{1}{u}\right)=H_2\left(\frac{1}{u}\right). \tag{1.11}$$

We note that the above equations are independent and may be treated separately, each of them being of the form of (1.1). A more complex problem leading to a

system of equations of the type of (1.1) was considered by Bartoshevich [1]. In [2] he developed a method for its solution which involved expansions in terms of Watson's operators. Sub-Sizonenko [19] provided an analytic $\mathcal{L}_2$ solution for (1.1) and other expressions valid in weighted $\mathcal{L}_p$ spaces were derived by Rooney [17] and Lamb ([11], [12]) but none is useful if a tabulated solution is required.
We note that $\int_0^t P(t,s)ds$ is divergent. Following [12] we use the transformations

$$y(t) = t^{-\mu} F(t), \qquad f(t) = t^{-\mu} H(t), \tag{1.12}$$

where $\mu > 0$ is a constant. From (1.1) we obtain

$$y(t) + \int_0^t q(t,s) y(s) ds = f(t), \quad t \in [0,T], \tag{1.13}$$

with

$$q(t,s) := \frac{1}{\sqrt{\pi}} \frac{1}{\sqrt{\ln(t/s)}} \left(\frac{s}{t}\right)^{\mu} \frac{1}{s}. \tag{1.14}$$

In [5] this equation has been transformed into the more tractable equation

$$y(t) - \int_0^t p(t,s) y(s) ds = g(t), \quad t \in [0,T], \tag{1.15}$$

with

$$p(t,s) := \left(\frac{s}{t}\right)^{\mu} \frac{1}{s}. \tag{1.16}$$

We have the following existence and uniqueness results.

Theorem 1 *For a given non-negative integer m, let $V_m[0,T]$ denote the normed space of the real functions ϕ such that $\phi \in C^m[0,T]$ with*

$$\|\phi\|_m := \max_{0 \le j \le m} \max_{t \in [0,T]} \left| \phi^{(j)}(t) \right|.$$

(i) [5] Let $\mu > 1$ in (1.14) and (1.16). If the function f in (1.13) belongs to V_m then the problem (1.13) − (1.14) possesses a unique solution $y \in V_m$. Similarly, if the function g in (1.15) belongs to V_m then (1.15) − (1.16) possesses a unique solution $y \in V_m$. Moreover, if g is given by

$$g(t) = f(t) - \int_0^t q(t,s) f(s) ds$$

then the solution of the problem (1.13) − (1.14) is also the solution of (1.15) − (1.16).

(ii) [8] Let $0 < \mu \le 1$ in (1.14) and (1.16). Then equations (1.13) − (1.14) and (1.15) − (1.16) have a family of solutions in $C[0,T]$ of which only one has C^1 continuity.

It should be noted that (i) is in contrast with the smoothness properties of weakly singular equations possessing a singularity in the kernel of the form $k(t,s) = (t-s)^{-\alpha}, 0 < \alpha < 1$ (Abel type equations). For those equations, a smooth forcing function leads to a solution which has typically unbounded derivatives at $t = 0$. Furthermore, unlike the function k, the functions q and p do not satisfy $\int_0^t q(t,s)ds \to 0$ as $t \to 0$ and $\int_0^t p(t,s)ds \to 0$ as $t \to 0$. In fact these integrals are constant and equal to $1/\sqrt{\mu}$ and $1/\mu$, respectively. Finally, all the iterated kernels associated with p (we note that q is the iterated kernel of second order) are unbounded. The ideas of iterated and discrete iterated kernels were used by Dixon and McKee [7] to obtain generalised weakly singular Gronwall inequalities (see also [3]). These are an important tool for proving convergence of discretization methods for weakly singular equations but it is usually required the existence of at least one bounded iterated kernel. Owing to the above properties, special techniques are needed to prove convergence of discretization methods for the above equations. Tang *et al* [20] applied the product Euler and Trapezoidal methods to equation (1.13)-(1.14) and obtained approximations to $y(t)$ of orders one and two, respectively. Diogo *et al* [5] considered a fourth order Hermite-type collocation method for (1.15)-(1.16) and Lima and Diogo [13] developed an extrapolation method, based on Euler's method. By introducing some appropriate function spaces they were also able to consider unbounded solutions. Recently, in [6], the construction of high order numerical methods for (1.15)-(1.16) has been investigated, under the assumption that the solution is sufficiently smooth. In this work we give a more detailed convergence analysis of those methods and show how the continuity requirements can be relaxed, in particular by employing the techniques developed in [13]. Related methods have been studied by several authors for Abel equations. Linz [14] considered a product integration method based on Simpson's rule and proved convergence of order three. By a sharper analysis, de Hoog and Weiss ([9],[10]) were able to prove that, for $\alpha = 1/2$, the convergence order was in fact 7/2. Cameron and McKee [4] extended the results of [10] to higher order methods.

This paper is organised as follows. In Section 2 a class of product integration methods based on the use of a main interpolatory quadrature rule together with several end rules is introduced. The consistency order of these methods is analysed in Section 3. In general, if the main rule is based on $n+1$ points, then consistency of order $n+1$ can be achieved if the solution $y(t)$ is in $C^{n+1}[0,T]$. However, if μ is sufficiently high, these continuity requirements can be relaxed. Following the approach used in [13], an appropriate transformation of the dependent variable is made so that the new equation possesses smooth solutions. The methods of Section 2 can then be applied to the transformed equation. Finally, sufficient conditions for product integration methods to be convergent are derived in Section 4. Two results were obtained; the former requires the sum of the modulus of the weights to be uniformly bounded by some constant less than one; the latter requires the weights to be non-negative. In Section 4 an example of a method corresponding to $n = 2$ is considered and its convergence properties discussed. The paper concludes with a sample of numerical examples illustrating the theoretical results obtained.

2. High order product integration methods

In order to construct numerical methods for (1.15)-(1.16), let us define the grid

$$\{t_j = jh, \quad 0 \leq j \leq N; Nh = T\}$$

and consider the discretised form of (1.15)

$$y(t_i) - \int_0^{t_i} p(t_i, s) y(s) ds = g(t_i), \quad 0 \leq i \leq N. \tag{2.1}$$

To approximate the integral in (2.1), some $(n+1)$-point interpolatory formula is used repeatedly, followed by an end rule or a series of end rules when necessary. The coefficients of the resulting quadrature are calculated analytically.

Suppose first that i is a multiple of n, say $i = nr$. We can rewrite (2.1) as

$$y(t_i) - \sum_{j=0}^{r-1} \int_{t_{nj}}^{t_{n(j+1)}} \left(\frac{s}{t_i}\right)^{\mu} \frac{1}{s} y(s)\ ds = g(t_i). \tag{2.2}$$

Approximating $y(s), s \in (t_j, t_{j+n})$ by a polynomial of degree n, yields

$$\int_{t_j}^{t_{j+n}} \left(\frac{s}{t_i}\right)^{\mu} \frac{1}{s}\ y(s) ds \simeq \sum_{k=0}^{n} y(t_{j+k}) \int_{t_j}^{t_{j+n}} \left(\frac{s}{t_i}\right)^{\mu} \frac{1}{s} l_k(s) ds, \tag{2.3}$$

where the l_k are the Lagrange polynomials of degree n associated with $t_j, t_{j+1}, \cdots,$ t_{j+n}, that is

$$l_k(s) = \prod_{\substack{i=0 \\ i \neq k}}^{n} \frac{s - t_{j+i}}{t_{j+k} - t_{j+i}} \quad 0 \leq k \leq n.$$

By making the transformation $s = t_j + vh$, we have

$$\begin{aligned} \int_{t_j}^{t_{j+n}} \left(\frac{s}{t_i}\right)^{\mu} \frac{1}{s} l_k(s) ds &= h \int_0^n \frac{(t_j + vh)^{\mu-1}}{(t_i)^{\mu}} l_k(t_j + vh) dv \\ &= \frac{1}{i^{\mu}} \int_0^n (j+v)^{\mu-1} \rho_k(v) dv, \end{aligned} \tag{2.4}$$

where ρ_k is a polynomial of degree n, given by

$$\rho_k(v) := l_k(t_j + vh) = \prod_{\substack{i=0 \\ i \neq k}}^{n} \frac{v - i}{k - i}. \tag{2.5}$$

Let us define

$$b_{\gamma}(j) := \int_0^n (j+v)^{\mu-1} \rho_{\gamma}(v) dv, \quad 0 \leq \gamma \leq n. \tag{2.6}$$

Substituting (2.4) into (2.3) and using (2.6), we obtain

$$\int_{t_j}^{t_{j+n}} \left(\frac{s}{t_i}\right)^{\mu} \frac{1}{s} y(s)ds \simeq \frac{1}{i^{\mu}} \sum_{k=0}^{n} y(t_{j+k}) b_k(j). \tag{2.7}$$

Combining (2.7) and (2.2), we obtain for the case $i = nr$ the following equation

$$y_{nr} - \frac{1}{(nr)^{\mu}} \sum_{j=0}^{r-1} \sum_{k=0}^{n} y_{nj+k}\; b_k(nj) = g(t_{nr}), \tag{2.8}$$

where the y_i are approximate values of $y(t_i)$.

When i is not a multiple of n, say $i = nr + \nu$, $1 \leq \nu \leq n-1$, it is convenient to rewrite (2.1) in the form

$$y(t_i) - \int_0^{t_{i-n-\nu}} p(t_i, s)y(s)ds - \int_{t_{i-n-\nu}}^{t_i} p(t_i, s)y(s)ds = g(t_i). \tag{2.9}$$

The integral over $[0, t_{i-n-\nu}]$ is approximated by the main repeated formula. In order to approximate the integral over $[t_{i-n-\nu}, t_i]$, end rules based on polynomial interpolation of degrees $(n+1), (n+2), \cdots, (2n-1)$ are used. Thus

$$\int_{t_{i-n-\nu}}^{t_i} p(t_i, s)y(s)ds \simeq \sum_{j=0}^{n+\nu} y(t_{i-n-\nu+j}) \int_{t_{i-n-\nu}}^{t_i} p(t_i, s)\tilde{l}_j(s)ds, \tag{2.10}$$

where the $\tilde{l}_j$ are the Lagrange polynomials of degree $n+\nu$ associated with the points $t_{i-n-\nu+j}$, $\quad 0 \leq j \leq n+\nu$. Making the transformation $s = t_{i-n-\nu} + vh$, we obtain

$$\int_{t_{i-n-\nu}}^{t_i} \left(\frac{s}{t_i}\right)^{\mu} \frac{1}{s}\tilde{l}_j(s)ds = \frac{1}{i^{\mu}} \int_0^{n+\nu} (i-n-\nu+v)^{\mu-1} \tilde{\rho}_j(v)dv, \tag{2.11}$$

where

$$\tilde{\rho}_j(v) := \tilde{l}_j(t_{i-n-\nu} + vh) = \prod_{\substack{k=0 \\ k\neq j}}^{n+\nu} \frac{v-k}{j-k}, \quad 0 \leq j \leq n+\nu. \tag{2.12}$$

Let us define

$$d_k^{\nu}(i) := \int_0^{n+\nu} (i-n-\nu+v)^{\mu-1} \tilde{\rho}_k(v)dv, \quad 0 \leq k \leq n+\nu. \tag{2.13}$$

Combining (2.10) and (2.11) and using (2.13) results in

$$\int_{t_{i-n-\nu}}^{t_i} p(t_i, s)y(s)ds \simeq \frac{1}{i^{\mu}} \sum_{k=0}^{n+\nu} y(t_{i-n-\nu+k})\; d_k^{\nu}(i). \tag{2.14}$$

Substituting (2.14) into (2.9) and treating the first integral in (2.9) as the one in (2.2), gives the following approximate equation for the case when $i = nr + \nu$

$$y_{nr+\nu} - \frac{1}{(nr+\nu)^{\mu}} \left(\sum_{j=0}^{r-2} \sum_{k=0}^{n} y_{nj+k}\ b_k(nj) + \sum_{k=0}^{n+\nu} y_{nr-n+k}\ d_k^{\nu}(nr+\nu) \right) = g(t_{nr+\nu}). \tag{2.15}$$

Combining (2.8) and (2.15), we obtain the discretization method

$$\mathbf{\Phi}_h \mathbf{y} = \mathbf{0} \tag{2.16}$$

with

$$[\mathbf{\Phi}_h \mathbf{y}]_i = \begin{cases} y_i - \tilde{y}_i, & 0 \le i \le n-1, \\ y_i - \sum_{j=0}^{i} \omega_{ij} y_j - g_i, & n \le i \le N, \end{cases} \tag{2.17}$$

where $\tilde{y}_i,\ 0 \le i \le n-1$, are given starting values. We note that, since $\mu > 1$, all the integrals involved can be calculated analytically.

3. Convergence

Let $\delta(h, t_i)$ denote the local consistency error, defined by

$$\delta(h, t_i) := \int_0^{t_i} \left(\frac{s}{t_i} \right)^{\mu} \frac{1}{s} y(s) ds - \sum_{j=0}^{i} \omega_{ij} y(t_j). \tag{3.1}$$

We require the following definition of consistency.

Definition 1 *The discretization* (2.16) − (2.17) *is said to be consistent of order* p *with* (1.15) − (1.16) *at* $t = t_i, i \ge n$, *if there exists a constant* C, *independent of* h, *such that for* $h \in (0, h_0), h_0 > 0$, *we have*

$$|\delta(h, t_i)| \le C h^p. \tag{3.2}$$

Theorem 2 *If* $y \in C^m[0,T], 0 < m \le n+1$, *then* $\delta(h, t_i) = O\left(h^m\right)$. *In particular, if* $y \in C^{n+1}[0,T]$ *then the discretization* (2.16) − (2.17) *is consistent of order* $n+1$.

When n is even, it is possible to have higher consistency order if y is sufficiently smooth.

Theorem 3 *Suppose that* n *is even. If* $y \in C^{n+2}[0,T]$ *and* $y^{(n+1)}(0) = 0$ *then* $\delta(h, t_i) = O(h^{n+2})$.

Non-smooth solutions

We have seen that in order to obtain consistency of order at least $n+1$, the function $y(t)$ was required to be at least in the class $C^{n+1}[0,T]$ (cf. Theorems 2 and 3). We now show that these continuity requirements can be relaxed if μ is sufficiently high. Following the approach used in [13], we can transform the original equation into a new equation, so that the two equations are equivalent away from the origin. Moreover, the corresponding solution of the new equation will be smooth. Let $\beta > 0$ be such that $\mu - \beta > 1$. Multiplying both sides of (1.15) by t^β and defining

$$\overline{y}(t) := t^\beta y(t), \tag{3.3}$$
$$\overline{g}(t) := t^\beta g(t) \tag{3.4}$$

we obtain the equation

$$\overline{y}(t) - \int_0^t \left(\frac{s}{t}\right)^{\mu-\beta} \frac{1}{s}\, \overline{y}(s)ds = \overline{g}(t), \tag{3.5}$$

which is equivalent to (1.15) for $t > 0$. If we apply the scheme (2.16)-(2.17) to this equation, we obtain approximations $\overline{y}_k$ of $\overline{y}(t_k) = t_k^\beta y(t_k)$, $k \geq n$. Then we take as an approximation to $y(t_k)$ the value $\overline{y}_k / t_k^\beta$. The idea is to chose β such that $t^\beta y(t)$ will have the required continuity. Let $\overline{\delta}(h, t_i)$ denote the consistency error associated with the scheme (2.16)-(2.17) applied to the transformed equation (3.5).

Theorem 4 *Let β be a real number such that $0 \leq \beta < \mu - 1$. If $t^\beta y(t) \in C^m[0,T]$, $0 < m \leq n+1$, then $\overline{\delta}(h, t_i) = O(h^m)$.*

Theorem 5 *Let $y(t) = t^\alpha\, f(t)$, with $f(t) \in C^{n+1}[0,T]$. If β is such that $0 \leq \beta < \mu - 1$ and $\alpha + \beta$ is an integer then $\overline{\delta}(h, t_i) = O(h^{n+1})$.*

Two convergence results

We start with the following Definition.

Definition 2 *The starting values are said to be convergent of order p if there exists a constant C_2, independent of h, such that*

$$|y(t_i) - y_i| \leq C_2 h^p, \quad 0 \leq i \leq n-1. \tag{3.6}$$

Theorem 6 *If the exact solution y of $(1.15)-(1.16)$ is such that the discretization $(2.16)-(2.17)$ is consistent of order $p (p \leq n+2)$, the starting values are convergent of order p and the weights ω_{ij} satisfy the condition*

$$\sum_{j=0}^{i-1} \frac{|\omega_{ij}|}{1 - |\omega_{ii}|} \leq \beta < 1, \quad i = n, n+1, \ldots. \tag{3.7}$$

then the scheme (2.16) − (2.17) *is convergent of order* p.

Proof: Let us define $e_i := y(t_i) - y_i$. We have

$$\begin{aligned} e_i &= \int_0^{t_i} p(t_i, s)y(s)ds - \sum_{j=0}^{i} \omega_{ij} y_j \\ &= \sum_{j=0}^{i} \omega_{ij}(y(t_j) - y_j) + \int_0^{t_i} p(t_i, s)y(s)ds - \sum_{j=0}^{i} \omega_{ij} y(t_j), \end{aligned} \tag{3.8}$$

which gives, after using (3.1) and taking modulus,

$$|e_i|\ (1 - |\omega_{ii}|) \le \sum_{j=0}^{i-1} ||\omega_{ij}||e_j| + |\delta(h, t_i)|. \tag{3.9}$$

It can be proved that

$$|\omega_{ii}| \le 1/\mu. \tag{3.10}$$

Then we obtain, for some constant C independent of h,

$$|e_i| \le \sum_{j=0}^{i-1} \frac{|\omega_{ij}|}{1 - |\omega_{ii}|}\ |e_j| + Ch^p, \tag{3.11}$$

where we have used (3.2). Finally, an application of a Gronwall Lemma [14] yields the desired result. ■

Theorem 7 *If the exact solution* y *of* (1.15) − (1.16) *is such that the discretization* (2.16) − (2.17) *is consistent of order* $p(p \le n+2)$, *the starting values are convergent of order* p *and the weights* ω_{ij} *are non-negative, then the scheme* (2.16) − (2.17) *is convergent of order* p.

Proof: Taking modulus in (3.8) and using (3.2) yields

$$|e_i| \le \sum_{j=0}^{i} |\omega_{ij}||e_j| + Ch^p. \tag{3.12}$$

It can be shown that

$$\sum_{j=0}^{i} \omega_{ij} = \frac{1}{\mu}. \tag{3.13}$$

Defining $E := \max_{0 \le j \le N}\{|e_j|\}$, it follows from (3.12) that

$$|e_i| \le \frac{1}{\mu} E + Ch^p, \quad n \le i \le N, \tag{3.14}$$

which then implies

$$E \leq \frac{E}{\mu} + Ch^p. \tag{3.15}$$

Since $\mu > 1$, (3.15) yields that

$$E \leq \frac{\mu}{\mu - 1} Ch^p \leq C_1 h^p. \tag{3.16}$$

■

4. The product Simpson's method

We consider the case when $n = 2$. To approximate the integral in (2.1), the product Simpson's rule is used repeatedly over $[0, t_i]$ if i is even. When i is odd, we use the product Simpson's rule over $[0, t_{i-3}]$ and the product three-eights rule will be used over $[t_{i-3}, t_i]$. In this case, (2.8) and (2.15) take the form, respectively,

$$y_{2r} - \frac{1}{(2r)^\mu} \sum_{j=0}^{r-1} \sum_{k=0}^{2} y_{2j+k} \; b_k(2j) = g(t_{2r}) \tag{4.1}$$

and

$$y_{2r+1} - \frac{1}{(2r+1)^\mu} \left(\sum_{j=0}^{r-2} \sum_{k=0}^{2} y_{2j+k} \; b_k(2j) + \sum_{k=0}^{3} y_{2r-2+k} \; d_k^1(2r+1) \right) = g(t_{2r+1}), \tag{4.2}$$

with the $b_k(j)$ and $d_k^1(i)$ defined by (2.6) and (2.13), respectively.

We note that, unlike in the Trapezoidal method (corresponding to the case $n = 1$), the condition $\omega_{ij} \geq 0$ does not hold for all i, j and all values of $\mu > 1$. Therefore, we cannot apply Theorem 7 for all values of μ. However we have the following result.

Theorem 8 *Let $1 < \mu \leq 2$. Consider the integrals $b_k(2r)$, $k = 0, 1, 2$, $r \geq 0$; and $d_l^1(2r + 1)$, $l = 0, 1, 2, 3$, $r \geq 1$. We have $b_0(2r) = 0$ if $r = 0$ and $\mu = 2$ and in the remaining cases all the integrals $b_k(2r)$ and $d_l^1(2r + 1)$ are positive. As a consequence, the weights ω_{ij} of algorithm (4.1) – (4.2) satisfy*

$$\omega_{ij} \geq 0, \quad i = 1, 2, \ldots; j = 0, 1, \ldots. \tag{4.3}$$

Assuming consistency, from Theorems 8 and 7 it follows immediately that Simpson's method is convergent if $1 < \mu \leq 2$. Convergence for the values $\mu > 2$ will be established with the help of Theorem 6 and this will require the following result.

Theorem 9 *Let $\mu > 1$. Then the weights ω_{ij} of algorithm* (4.1) − (4.2) *satisfy*

$$(a) \quad \sum_{j=0}^{i} |\omega_{ij}| \le \begin{cases} \dfrac{\lambda_2}{\mu}, & i = 2r, \\ \\ \dfrac{\lambda_3}{\mu}, & i = 2r+1, \quad r \ge 1, \end{cases} \tag{4.4}$$

$$(b) \quad 0 < \omega_{ii} \le \frac{\mu}{\mu^2 + 3\mu + 2}, \quad i = 2, 3, \ldots \tag{4.5}$$

Above $\lambda_2 = 1.25$ and $\lambda_3 = 1.63113$ are the Lebesgue constants associated with the Lagrange polynomials of second and third degrees, respectively (see e.g. [16]).

We are now in a position to apply Theorem 6 in the case $\mu > 2$. Indeed, from (4.4) and (4.5) it follows that

$$\sum_{j=0}^{i-1} \frac{|\omega_{ij}|}{1 - \omega_{ii}} \le \frac{1.63113}{\mu} \, \frac{\mu^2 + 3\mu + 2}{\mu^2 + 2\mu + 2}, \quad i = 2, 3, \ldots . \tag{4.6}$$

That is, condition (3.7) is satisfied with $\rho = 1.63113\,(\mu^2{+}3\mu{+}2)/(\mu^3{+}2\mu^2{+}2\mu) < 1$, since $\mu > 2$. This completes the proof of convergence (of order p) of the product Simpson's method for all values of $\mu > 1$, assuming that it is consistent of order p and that the starting values are convergent of the same order.

Remark 1 By similar arguments to the ones employed in the proof of Theorem 8, it can be shown that the weights of the discretization method (2.16)-(2.17) in the case $n = 1$ are non-negative for all values of μ; in the cases $n = 3, n = 4$ the weights are non-negative for the values $1 < \mu \le 2$. Therefore, convergence of the methods obtained with $n = 1, 3, 4$ is assured for the corresponding values of μ.

5. Numerical results

In order to illustrate the theoretical results of the previous Sections we have considered the numerical solution of equation (1.15)-(1.16), with $g(t) = t^\alpha\ (\mu - 1)/\mu$ and exact solution $y(t) = t^\alpha$, $t \in [0, 2]$. The following choices of α and μ have been considered.

Example 1: $\mu = 1.5$ and $\alpha = 6.5$;

Example 2: $\mu = 1.3$ and $\alpha = 3.0$;

Example 3: $\mu = 5.0$ and $\alpha = -0.2$.

In Tables 1–5 the errors $|e_i| = |y(t_i) - y_i|$ produced with several numerical methods of the type (2.16)-(2.17) are displayed for the cases $n = 2, 3, 4$.

Table 1 contains the errors obtained with the product Simpson's method for Example 1. In this case Theorem 3 can be applied with $n = 2$, giving consistency of order four. The results indicate that the convergence order is four and this is in agreement with the theoretical prediction.

To obtain the values in Table 2, we implemented a method based on the repeated use of the product three-eights rule as the main rule (three eight's method) which was applied to Example 1. Here Theorem 2 can be applied with $m = n+1 = 4$ and the results show the expected fourth order of convergence (cf. Remark 1).

A further method corresponding to $n = 4$ was also implemented and applied to Example 1. Here $y^{(5)}(t) = \text{constant} \times t^{1.5}$, so that $y^{(5)}(0) = 0$ and Theorem 3 can be applied. This together with Remark 1 gives convergence of order six which is confirmed by the numerical results in Table 3.

We have also applied Simpson's method to Example 2 for which $y^{(3)}(t) \equiv \text{const} \neq 0$. We can only conclude that the method is consistent of order three and we would also expect to get convergence of order three. Here the numerical results of Table 4 seem to indicate that the convergence order might be slightly higher, maybe 3.3. We note that this is just $\alpha + \mu - 1$. Further numerical results for other values of μ seem to support the conjecture that the order should be given by $\min(\alpha + \mu - 1, n + 2)$.

In Table 5 the results obtained for Example 3 by Simpson's method are displayed. In order to deal with the unbounded solution $y(t) = t^{-0.2}$ we considered the transformed equation (3.5) with β satisfying the conditions $\mu - \beta > 1$ and $t^\beta y(t) \in C^3[0, T]$. As $\mu = 5$ we chose $\beta = 3.6$ so that the solution of the new equation (3.5) was $t^\beta y(t) = t^{3.4}$. Theorem 4 can be applied to equation (3.5) giving third order of convergence. Based on the numerical results obtained, we conjecture that, in general, the order, let us say, q, of a method should be given by: $q = \min(\alpha + \mu - 1, n + 1)$ if n is odd; $q = \min(\alpha + \mu - 1, n + 2)$ if n is even.

Table 1: *The Simpson's method*
Example 1: $\mu = 1.5, \alpha = 6.5$

t_i	0.5	1.0	2.0	rate
$h = 0.05$	$1.9D-6$	$1.1D-5$	$5.9D-5$	
$h = 0.025$	$1.2D-7$	$6.6D-7$	$3.7D-6$	4.0
$h = 0.0125$	$7.2D-9$	$4.1D-8$	$2.3D-7$	4.0
$h = 0.00625$	$4.5D-10$	$2.5D-9$	$1.4D-8$	4.0

Table 2: *The three-eights method*
Example 1 : $\mu = 1.5, \alpha = 6.5$

t_i	0.5	1.0	2.0	rate
$h = 0.05$	$5.8D-7$	$5.1D-6$	$5.4D-5$	
$h = 0.025$	$5.6D-8$	$6.0D-7$	$4.0D-6$	3.7
$h = 0.0125$	$6.6D-9$	$4.4D-8$	$2.9D-7$	3.8
$h = 0.00625$	$4.9D-10$	$3.2D-9$	$1.9D-8$	3.9

Table 3: *A method based on a five-point rule rule*
Example 1 : $\mu = 1.5, \alpha = 6.5$

t_i	0.5	1.0	2.0	rate
$h = 0.05$	$3.8D-8$	$1.54D-7$	$2.4D-7$	
$h = 0.025$	$1.7D-9$	$2.6D-9$	$4.0D-9$	5.9
$h = 0.0125$	$2.9D-11$	$4.5D-11$	$6.7D-11$	5.9
$h = 0.00625$	$4.9D-13$	$7.4D-13$	$1.1D-13$	6.0

Table 4: *The Simpson's method*
Example 2: $\mu = 1.3, \alpha = 3.0$

t_i	0.5	1.0	2.0	rate
$h = 0.05$	$3.8D-6$	$3.3D-6$	$2.8D-6$	
$h = 0.025$	$4.1D-7$	$3.5D-7$	$2.9D-7$	3.26
$h = 0.0125$	$4.3D-8$	$3.6D-8$	$3.0D-8$	3.30
$h = 0.00625$	$4.5D-9$	$3.7D-9$	$3.0D-9$	3.30

Table 5: *The Simpson's method*
Example 3: $\mu = 5.0, \alpha = -0.2, \beta = 3.6$

t_i	0.5	1.0	2.0	rate
$h = 0.05$	$4.1D-6$	$3.6D-6$	$3.1D-6$	
$h = 0.025$	$3.5D-7$	$3.0D-7$	$2.5D-7$	3.6
$h = 0.0125$	$2.8D-8$	$2.4D-8$	$1.9D-8$	3.7
$h = 0.00625$	$2.2D-9$	$1.8D-9$	$1.5D-9$	3.7

6. Concluding remarks

A class of product integration methods based on $n+1$-point interpolatory rules has been introduced for the solution of (1.15)-(1.16). A sufficient condition for convergence was derived which required the weights to satisfy (3.7) (cf. Theorem 6). For the trapezoidal method (n=1) it is easy to prove that this condition holds for all values of $\mu > 1$. In general, for methods based on rules using $2, 3, 4$ points we conjecture that

$$\sum_{j=0}^{i-1} \frac{|\omega_{ij}|}{1 - |\omega_{ii}|} \le 1/\mu, \quad i = n, n+1, \ldots . \tag{6.1}$$

Another condition for convergence was provided by Theorem 7 which required the weights to be non-negative. This again can be easily shown to be true for the Trapezoidal method for any μ. However, for a discretization method of the type (2.16)-(2.17) with $n > 1$, the sign of the weights will depend on μ. The convergence of numerical methods can be based on a combination of Theorems 6 and 7 for adequate values of μ. In the case of Simpson's method, it was proved that the weights were non-negative if $1 < \mu \le 2$, while a condition like (3.7) was shown to hold if $\mu > 2$.

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