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A computational program for the Newton polyhedron and the integral closure of ideals

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Abstract We describe the implementation of a computational program that, for a given ideal with finite codimension in the ring of complex polynomials, constructs the Newton polyhedron and determines the integral closure of the ideal. We follow the definition given in [5] for the construction of the Newton polyhedron and the algorithm described in [6] for the computation of the integral closure. This program is divided in two modules, one for the calculation in the ring $\mathbb{C}[x,y]$ and another that works in the ring $\mathbb{C}[x,y,z]$.

1. Introduction

The integral dependence relation of ideals is one of the main tools in the study of algebraic, geometrical and topological incidence relations between germ of functions $f:\mathbb{C}^n,0\to\mathbb{C},0$.

Saia in [5] showed the relationship between the integral closure and the Newton non degeneracy condition of an ideal, Saia in [6] gave an algorithm for the determination of the integral closure of any ideal of finite codimension in the ring of convergent power series $\mathbb{C}\{x_1,\ldots,x_n\}$. This algorithm is based in the construction of the toroidal embedding associated to the Newton polyhedron of the ideal.

We describe here the implementation of these algorithms in a computational program that works in the ring of polynomials in two (or three) variables, denoted by $\mathbb{C}[x,y]$ ($\mathbb{C}[x,y,z]$). In the module for two variables, we implemented three short subprograms for the calculation of the integral closure of the jacobian ideal $J(g)=\left\langle \frac{\partial f}{\partial x},\frac{\partial f}{\partial y}\right\rangle$ and the associated ideals $\left\langle x\frac{\partial f}{\partial x},y\frac{\partial f}{\partial y}\right\rangle$, $\left\langle x\frac{\partial f}{\partial x},y\frac{\partial f}{\partial x},y\frac{\partial f}{\partial y},x\frac{\partial f}{\partial y}\right\rangle$ for

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any germ of polynomial $f \in \mathbb{C}[x, y]$ which has isolated singularity at 0. We see in the works of Yoshinaga [8], Ruas & Saia [4], and Kouchnirenko [3], that the integral closure of such ideals is the key tool to calculate topological and geometrical invariants of germs of functions.

This program is the first part of a package that is being implemented for the determination of geometrical and topological invariants of germs of isolated singularity mappings $f{:}\mathbb{C}[x_1, \ldots, x_n] \to \mathbb{C}[y_1, \ldots, y_p]$ with $n, p \leq 3$. The program works in Windows 95, is written in Pascal with Delphi compiler and the three dimensional module uses the Graphics interface OpenGL.

2. Theoric results

We resume here definitions and results given in [5] and [6] needed for the implementation of the software.

2.1. Newton polyhedron and the non-degeneracy condition

Let $\mathbb{C}\{x_1, \ldots, x_n\}$ be the ring of convergent power series around the origin with a fixed coordinate system $\mathbf{x} = (x_1, \ldots, x_n)$. For each series $g(x) = \sum a_k x^k$, we define *supp* $g = \{k \in \mathbf{Z}^n : a_k \neq 0\}$. For any ideal I in $\mathbb{C}\{x\}$, we call *supp* $I = \cup \{supp\ g{:}\, g \in I\}$.

Definition 1 *The Newton polyhedron of I, denoted by $\Gamma_+(I)$, is the convex hull in $\mathbb{R}^n_+$ of the set*

$$\cup \left\{k + v : k \in supp\ I,\ v \in \mathbb{R}^n_+\right\}.$$

We denote by $\Gamma(I)$, the union of all compact faces of $\Gamma_+(I)$.

In the sequel we shall consider $I = \langle g_1, g_2, \ldots, g_s \rangle$ an ideal of finite codimension in $\mathbb{C}\{x\}$, i.e., $dim_{\mathbb{C}} \frac{\mathbb{C}\{x\}}{\langle g_1, g_2, \ldots, g_s \rangle} < \infty$.

Given a finite subset $\Delta \subset \Gamma_+(I)$, for any series $g(x) = \sum a_k x^k \in \mathbb{C}\{\mathbf{x}\}$ we call $g_\Delta = \sum_{k \in \Delta} a_k x^k$. We denote by $C(\Delta)$ the cone of half-rays emanating from 0 and passing through Δ. Since $C(\Delta) \cap \mathbb{Z}^k$ is a subsemigroup of $\mathbb{Z}^k$, the subset of $\mathbb{C}\{x\}$ given by $C_\Delta = \{g \in \mathbb{C}\{x\} : supp\ g \subseteq C(\Delta) \cap \mathbb{Z}^k\}$ is a subring with unity of $\mathbb{C}\{x\}$.

Definition 2 Algebraic characterization of non-degeneracy
A subset $\Delta \subset \Gamma_+(I)$ is non-degenerate if the ideal I_Δ generated by $\{g_{1_\Delta}, g_{2_\Delta}, \ldots, g_{s_\Delta}\}$ has finite codimension in C_Δ.

The above definition is equivalent to the following:

Geometric characterization of non-degeneracy
Δ is non-degenerate if the equations $g_{1_\Delta}(x) = \ldots = g_{s_\Delta}(x) = 0$ have no common solution in $(\mathbb{C} - \{0\})^n$.

Definition 3 *An ideal I is Newton non-degenerate if all compact faces $F_i \subset \Gamma(I)$ are non-degenerate.*

Remark 1 *The collection of all cones $C(F_i)$ for all compact faces F_i of $\Gamma_+(I)$ gives a polyhedral decomposition for $\mathbb{R}^n_+$, since I is an ideal of finite codimension in $\mathbb{C}\{x\}$ and $\Gamma_+(I)$ is a convex polyhedron in $\mathbb{R}^n_+$.*

2.2. The integral closure and non-degenerate sets

Definition 4 *Let I be an ideal in a ring A, the integral closure of I, denoted by $\overline{I}$ is the ideal of the elements $h \in A$ that satisfies an integral dependence relation $h^n + a_1 h^{n-1} + \ldots + a_n = 0$ with $a_i \in I^i$.*

Proposition 1 [7] *When $A = \mathbb{C}\{x_1, \ldots, x_n\}$ the following statements are equivalent:*

1. *$h \in \overline{I}$;*
2. **(Growth condition)** *For each choice of generators $\{g_i\}$ of I there exists a neighbourhood U of x_0 and a constant $\varepsilon > 0$ such that for all $x \in U$:*

$$|| \ h(x) \ || \ \leq \ \varepsilon.sup||g_i(x)||;$$

3. **(Valuative criterion)** *For each analytic curve $\varphi: (\mathbb{C}, 0) \to (X, x_0)$, $h \circ \varphi$ lies in $(\varphi^*(I))\,(\mathbb{C}\{x\})$.*

These equivalences are essential in the algorithm that calculates the integral closure.

Definition 5 *We denote by $C(\overline{I})$ the convex hull in $\mathbb{R}^n_+$ of the set $\cup \left\{m : x^m \in \overline{I}\right\}$.*

Theorem 1 [5] *$C(\overline{I}) \subset \Gamma_+(I)$ and $C(\overline{I}) = \Gamma_+(I)$ if and only if I is Newton non-degenerate.*

When $\Gamma_+(I)$ has compact faces which are degenerate, we need the following results to compute the polyhedron $C(\overline{I})$.

We consider the dual space $\mathbb{R}^{n*}$ of $\mathbb{R}^n$ with coordinates $(a_1, \ldots, a_n)$.

Definition 6 *For each $a = (a_1, \ldots, a_n) \in \mathbb{R}^{n*}$ we let:*

a. $\ell(a) = min\{\langle a, k\rangle : k \in \Gamma_+(g_{i,j})\}$, $\langle a, k\rangle = \sum_{i=1}^{n} a_i k_i$;

b. $\Delta(a) = \{k \in \Gamma_+(g_{i,j}) : \langle a, k\rangle = \ell(a)\}$;

c. *Two vectors a and a' are equivalent if $\Delta(a) = \Delta(a')$.*

This equivalence relation gives a partition Σ of the positive octant of the dual space $\mathbb{R}^{n*}$ of $\mathbb{R}^n$ in a finite number of closed convex cones with their vertices at zero. This partition can be seen as the dual of the polyhedral decomposition of $\mathbb{R}^n_+$ given by the Newton polyhedron $\Gamma_+(I)$, given in Remark 1.

From this duality we see that for each $(n-1)$-dimensional face F_i in $\Gamma_+(I)$ there exists a class of integers vectors $a^i \in (\mathbb{Z}^n_+ - \{0\})$ such that $F_i = \Delta(a)$. These integers vectors are all normal vectors of the hyperplane that contains the face F_i, we denote by a^i a representative of this class of integers vectors and by $F(a^i)$ the corresponding face.

Definition 7 *For a compact face $F(a^i) \in \Gamma(I)$ and a given $\ell \in \mathbb{Z}_+$, we call*

$$F_\ell(a^i) = \{m \in \Gamma_+(I) \colon < m, a^i > \leq \ell\}.$$

It follows from the finite codimension of the ideal I that for any compact face $F(a^i) \subset \Gamma(I)$, there exists an level ℓ such that $F_\ell(a^i)$ is a non-degenerate set, hence, for each $(n-1)$-dimensional compact face $F(a^i) \subset \Gamma(I)$ we let:

Definition 8 $Q_i = min.\{\ell : \; F_\ell(a^i) \text{ is non} - \text{degenerate}\}.$

We denote by F_{Q_i} the first non-degenerate set of the compact face $F(a^i)$.

Theorem 2 [6] *For each $m = (m_1, \dots, m_n) \in \mathbb{Z}^n_+$ the following statements are equivalent:*

1. *$m \in C(\overline{I})$;*
2. *The inequality $Q_{a^i} \leq \langle m, a^i \rangle$ holds for each $(n-1)$-dimensional compact face $F(a^i) \subset \Gamma(I)$.*

Therefore we see that the computation of the first non-degenerate sets F_{Q_i} of all compact faces $F(a^i)$ of $\Gamma(I)$ is the key to compute $C(\overline{I})$.

The equivalence between the algebraic and geometric characterizations of non-degeneracy is the first tool that enable us calculate the non-degenerate sets $F_Q(a)$ in $\Gamma_+(I)$. Another tool to compute the numbers Q_i is the construction of the toroidal embedding of $\mathbb{C}^n$ induced by $\Gamma_+(I)$, we shall resume this construction here.

2.3. Toroidal embedding

From the duality between the partition Σ of $\mathbb{R}^{n*}$ and the polyhedral decomposition of $\mathbb{R}^n_+$ induced by $\Gamma_+(I)$, we have that each vertex v of $\Gamma(I)$ is a vertex of at least one set of n compact faces $F(a^i)$ of dimension $n-1$ in $\Gamma(I)$. For each such set of n compact faces such that v is a vertex of F_i we have an associated n-dimensional cone $\sigma \in \Sigma$ which is generated by the set of n integer vectors denoted by $\{a^1_\sigma, a^2_\sigma, \dots, a^n_\sigma\}$.

For each n-dimensional cone $\sigma = \sigma\left(a^1_\sigma, a^2_\sigma, \dots, a^n_\sigma\right)$ we define a mapping

$$\pi_\sigma \colon \mathbb{C}^n_\sigma \to \mathbb{C}^n : \; \pi_\sigma(y_1, \dots, y_n) = (y_1^{a_1^1} \cdot \ldots \cdot y_n^{a_1^n}, \dots, y_1^{a_n^1} \cdot \ldots \cdot y_n^{a_n^n}),$$

where $\left(a_1^j, \ldots, a_n^j\right)$ denotes the coordinates of the vector a_σ^j.

We glue two copies C_σ^n and $\mathbb{C}_\tau^n$ via the following equivalence relation, let $y_\sigma \in \mathbb{C}_\sigma^n$ and $y_\tau \in \mathbb{C}_\tau^n$, then $y_\sigma \sim y_\tau$ if and only if $\pi_\sigma(y_\sigma) = \pi_\tau(y_\tau)$. We denote the quotient set by $X = \{\sqcup\, \mathbb{C}^n(\sigma)\}/\sim$, where $\sqcup\, \mathbb{C}_\sigma^n$ is the disjoint union of all $\mathbb{C}_\sigma^n$.

X is a nonsingular n-dimensional algebraic complex manifold, called the *Toric Variety of* $\Gamma_+(I)$ and the application $\pi\colon X \to \mathbb{C}^n$ defined by $\pi(y) = \pi_\sigma(y_\sigma)$, is the toroidal embedding associated to $\Gamma_+(I)$.

Therefore, for any monomial $x^m \in \mathbb{C}^n$ and any n-dimensional cone $\sigma \in \Sigma$ the following conditions are equivalent:

(*) $|x^m| \leq \mathcal{E}.sup_i\ \{|g_i(x)|\}$ for all x in a neighbourhood U of 0;

(**) $|x^m| \circ \pi_\sigma(y_\sigma) \leq \mathcal{E}.sup_i\ \{|g_i| \circ \pi_\sigma(y_\sigma)\}$ for all $y_\sigma \in \pi^{-1}(U)$.

This equivalence gives us an efficient way to find the numbers Q_i associated to the first non-degenerate sets F_{Q_i}.

3. The algorithm

Algorithm MAIN

1. Enter with the POLYNOMIALS GENERATORS of the ideal;
2. If there are POLYNOMIALS GENERATORS

1. for all POLYNOMIALS GENERATORS; format the characters of the POLYNOMIALS GENERATORS;

2. If the POLYNOMIAL GENERATOR exceeds the memory; Show an error message and stop

3. If the POLYNOMIAL GENERATOR is Ok; them insert it in the LIST OF POLYNOMIALS and Execute algorithms GRAPHICS and REPORT

If not Show an error message and stop;

Algorithm GRAPHICS:

1. Find the first face of the Newton polyhedron and put it in the LIST OF FACES
2. For each edge of a face in the LIST OF FACES, find the adjacent face and include it in the LIST OF FACES
3. Plot the convex hull from the LIST OF FACES

Algorithm REPORT:

1. First “simple” test of finite codimension of the ideal: the generators must have monomials which are in the coordinate axes;

If the codimension is not finite, show an error message and finish;

2. Show the generators of the ideal, the faces of the Newton polyhedron and the normal vectors of each face;
3. Calculate all mappings π_σ of the toroidal embedding;
4. For each generator g_j, do the compositions $g_j \circ \pi_{F_i}$ for all mappings π_σ;

5. For each generator g_j do the restriction $g_{j_{|F_i}}$ for all compact faces F_i of $\Gamma(I)$;

6. For each generator g_j compute the functions $g_{j_{|F_i}} \circ \pi_\sigma$ for all $(n-1)$-dimensional compact faces F_i of $\Gamma(I)$;

7. For each compact face F_i of $\Gamma(I)$, test if there exists a non degenerate set associated to F_i;

8. If test **7.** is OK for all compact faces, do
  1. Execute the algorithm INTEGRAL CLOSURE;
  2. Show the monomials which are in the polyhedron $C(\overline{I})$;
if not show message of error and finish;

Algorithm INTEGRAL CLOSURE:

1. For each $(n-1)$-dimensional compact face of $\Gamma_+(I)$ do
  1. Find the first non-degenerate set associated to the face and include it in the LIST OF FACES;
  2. Plot the LIST OF FACES.

3.1. Construction of the Newton polyhedron

The construction of the Newton polygon of an ideal in $\mathbb{C}[x, y]$ is an adaptation of the algorithm of Jarvis for the construction of a convex hull of a finite set of points in the plane. In the three dimensional case this construction is an adaptation of the algorithm of Chand-Kapur. The algorithms of Jarvis and Chand-Kapur are described in [2], we show here the algorithm for the construction of the Newton polyhedron of an ideal with finite codimension in $\mathbb{C}[x, y, z]$.

Since the Newton polyhedra considered here are associated to ideals of finite codimension, a necessary condition to start the program is that there exists at least one point of the Newton polyhedron in each coordinate axis. This computation is done in the first "simple" test of the codimension, item 1. of the algorithm REPORT.

There are two types of faces in any Newton polyhedron, the coordinate faces (which are not compact) and a finite number of compact faces which are not in the coordinate planes.

The algorithm to construct these faces is the following:

First step: Determination of an initial face of the Newton polyhedron.

The initial face to be determined is the Newton polygon defined by the points which are in the plane xy.

Second step: Determination of an adjacent face of a fixed face.

Let F_i be a face, and α an edge of F_i which does not have the next adjacent face determined. The next adjacent face of this edge is the one which defines a maximal angle between the semi-planes defined by all the possible faces which have the edge α. We obtain the maximal angle checking the angle between the normal vectors of these possible faces.

If there are more than one "face" in this condition, these "faces" are part of a bigger face in a plane, hence we calculate this kind of face as a Newton polygon in this plane.
Final step: The process stops when we find all adjacent faces for all edges of all compact faces.

4. Determination of the non-degenerate sets

The main purpose of the algorithm REPORT is to check the existence of all non-degenerate sets associated to the compact faces. For this we use the associated mappings π_σ of the toroidal embedding in order to compute the numbers Q_i of each compact face F_i.

The purpose of the algorithm INTEGRAL CLOSURE is to find the first non-degenerate set F_{Q_i} of each $(n-1)$-dimensional compact face F_i of $\Gamma_+(I)$ and construct the polyhedron $C(\overline{I})$.

We describe now the algorithm used in the item 7. of the algorithm REPORT.

Test of existence of a non-degenerate set for each compact face F_i.

The main idea in the Theorem 2 is that for each compact face (n-1)-dimensional F_i of $\Gamma(I)$ and for each generator g_j of I, the first non-degenerate set F_{Q_i} gives the smallest polynomial $g_{j|F_{Q_i}}$ in the Taylor series of g_j, such that the set of common zeros of the equations $g_{j|F_{Q_i}} = 0$ is a subset of $(\mathbb{C} - \{0\})^n$.

We fix a face F_i and write each polynomial $G_{j,i} = g_{j|F_{Q_i}}$ in the form $G_{j,i} = x^a y^b z^c H_{j,i}(x, y, z)$ such that the zero set of $H_{j,i}$ is not in $\{\mathbb{C} - 0\}^n$. Then we compute if there are common solutions of the equations $H_{j,i} = 0$ in the following way: Consider the ideal H_i generated by $\{H_{1,i}, \ldots, H_{s,i}\}$ and apply the theorem bellow to the affine variety $V = V(H_i)$.

Theorem 3 [1] *Let $V = V(I)$ be an affine veriety in $\mathbb{C}^n$. For a fixed monomial order on $\mathbb{C}[x_1, \ldots, x_n]$ we have the equivalences:*

1. *V is a finite set;*
2. *For each $1 \leq i \leq n$ there exists one $m_i \geq 0$ such that $x_i^{m_i} \in \langle LT(I) \rangle$, where $\langle LT(I) \rangle$ denotes the ideal generated by the leading terms of I;*
3. *If G denotes a Grobner basis for I, then for all i, $1 \leq i \leq n$ there exists one $x_i^{m_i} = LM(g)$ for some $g \in G$, $LM(g)$ denotes the leading monomial of g;*
4. *The complex vector space $S = Span(x^\alpha : x^\alpha \notin \langle LT(I) \rangle)$ is of finite codimension;*
5. *The complex vector space $\frac{\mathbb{C}[x_1,\ldots,x_n]}{I}$ is of finite codimension.*

We remark that we are considering the equivalence between items 3 and 5 and the fact that if the vector space $\frac{\mathbb{C}[x_1,\ldots,x_n]}{I}$ is of finite codimension, them there are no common solutions for the equations $\{H_{1,i} = 0, \ldots, H_{s,i} = 0\}$ given by the generators of the ideal H_i.

Algorithm:

1. For each $(n-1)$-dimensional compact face F_i consider the restrictions of the generators g_j to this face and compute the correspondent polynomials $H_{j,i}$;

Loop: 2. Find the Grobner basis of the ideal $H_i = \langle H_{1,i}, \ldots, H_{s,i} \rangle$;

3. Apply the theorem above to determine if the set is or not non-degenerate;

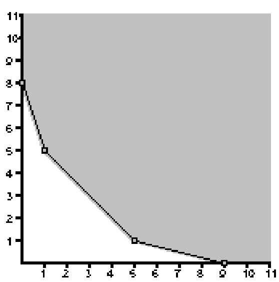
4. If the set is non-degenerate, go to the next face.
If not, search for the next level and add the set of monomials which are in this level to the list POLYNOMIALS GENERATORS and returns to Loop;

5. Continue this Loop until find the first non-degenerate set;

If for one compact face this set is not found, the ideal is not of finite codimension, therefore the algorithm does not compute the integral closure and shows message of error.

5. An example in $\mathbb{C}[x, y]$

Let $I = \langle g_1, g_2 \rangle$, with $g_1(x, y) = x^9 + x^5y - 2x^3y^3 + xy^5$, $g_2(x, y) = y^8 + x^5y - 2x^3y^3 + xy^5$



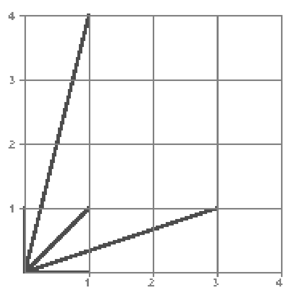


Figure 1: Newton polyhedron $\Gamma_+(I)$ Normal vectors a^i

Report

Generators of the ideal: $g_1(X, Y) = X^9 + X^5.Y - 2.X^3.Y^3 + X.Y^5$,
$g_2(X, Y) = Y^8 + X^5.Y - 2.X^3.Y^3 + X.Y^5$

Edges of the Newton polygon: $F_1 = \{[0, \infty], [0, 8]\}$, $F_2 = \{[0, 8], [1, 5]\}$,
$F_3 = \{[1, 5], [5, 1]\}$, $F_4 = \{[5, 1], [9, 0]\}$, $F_5 = \{[9, 0], [\infty, 0]\}$

Normal vectors: $a^1 = (1, 0)$, $a^2 = (3, 1)$, $a^3 = (1, 1)$, $a^4 = (1, 4)$, $a^5 = (0, 1)$

Mappings π_σ of the toroidal embedding:

$$\pi_{1,2}(y_1, y_2) = (y_1^1.y_2^3, y_1^0.y_2^1)$$
$$\pi_{2,3}(y_2, y_3) = (y_2^3.y_3^1, y_2^1.y_3^1)$$
$$\pi_{3,4}(y_3, y_4) = (y_3^1.y_4^1, y_3^1.y_4^4)$$
$$\pi_{4,5}(y_4, y_5) = (y_4^1.y_5^0, y_4^4.y_5^1)$$

Composition of the functions π with the generators of the ideal:

Generator g_1:

$$g_1 \circ \pi_{1,2}(y_1, y_2) = y_1.y_2^8 - 2.y_1^3.y_2^{12} + y_1^5.y_2^{16} + y_1^9.y_2^{27}$$

$$g_1 \circ \pi_{2,3}(y_2, y_3) = y_2^8.y_3^6 - 2.y_2^{12}.y_3^6 + y_2^{16}.y_3^6 + y_2^{27}.y_3^9$$

$$g_1 \circ \pi_{3,4}(y_3, y_4) = y_3^6.y_4^{21} - 2.y_3^6.y_4^{15} + y_3^6.y_4^9 + y_3^9.y_4^9$$

$$g_1 \circ \pi_{4,5}(y_4, y_5) = y_4^{21}.y_5^5 - 2.y_4^{15}.y_5^3 + y_4^9.y_5 + y_4^9$$

Generator g_2:

$$g_2 \circ \pi_{1,2}(y_1, y_2) = y_2^8 + y_1.y_2^8 - 2.y_1^3.y_2^{12} + y_1^5.y_2^{16}$$

$$g_2 \circ \pi_{2,3}(y_2, y_3) = y_2^8.y_3^8 + y_2^8.y_3^6 - 2.y_2^{12}.y_3^6 + y_2^{16}.y_3^6$$

$$g_2 \circ \pi_{3,4}(y_3, y_4) = y_3^8.y_4^{32} + y_3^6.y_4^{21} - 2.y_3^6.y_4^{15} + y_3^6.y_4^9$$

$$g_2 \circ \pi_{4,5}(y_4, y_5) = y_4^{32}.y_5^8 + y_4^{21}.y_5^5 - 2.y_4^{15}.y_5^3 + y_4^9.y_5$$

Restriction of the generators to the compact faces:

Generator g_1: $g_{1_{|F2}}(X, Y) = X.Y^5$
$g_{1_{|F3}}(X, Y) = X.Y^5 - 2.X^3.Y^3 + X^5.Y,$
$g_{1_{|F4}}(X, Y) = X^5.Y + X^9$

Generator g_2: $g_{2_{|F2}}(X, Y) = Y^8 + X.Y^5$
$g_{2_{|F3}}(X, Y) = X.Y^5 - 2.X^3.Y^3 + X^5.Y$
$g_{2_{|F4}}(X, Y) = X^5.Y$

Test of existence of non-degenerate sets of the compact faces

Compact face F_2:
$g_{1_{|F2}} \circ \pi_{2,3}(y_2, y_3) = y_2^8.y_3^6 = (y_2^8.y_3^6).(1)$
$g_{2_{|F2}} \circ \pi_{2,3}(y_2, y_3) = y_2^8.y_3^8 + y_2^8.y_3^6 = (y_2^8.y_3^6).(y_3^2 + 1)$
Compact face F_2 is non-degenerate
Vertices of the highest face of the non-degenerate set: $\{[0, 8.00]), \ [1.00, 0]\}$
Compact face F_3:
$g_{1_{|F3}} \circ \pi_{3,4}(y_3, y_4) = y_3^6.y_4^{21} - 2.y_3^6.y_4^{15} + y_3^6.y_4^9 = (y_3^6.y_4^9).(y_4^{12} - 2.y_4^6 + 1)$
$g_{2_{|F3}} \circ \pi_{3,4}(y_3, y_4) = y_3^6.y_4^{21} - 2.y_3^6.y_4^{15} + y_3^6.y_4^9 = (y_3^6.y_4^9).(y_4^{12} - 2.y_4^6 + 1)$
Compact face F_3 is degenerate
Searching for the non-degenerate set of this face.
Finding the polynomials of the next set:
Polynomial generator: $g_{1_{|F3}}(X, Y) = 0$
Polynomial generator: $g_{2_{|}F3}(X, Y) = Y^8$

Weighted polynomial: $g_{1_{|F3}} \circ \pi_{3,4}(y_3, y_4) = (y_3^0.y_4^0).(0)$
Weighted polynomial: $g_{2_{|F3}} \circ \pi_{3,4}(y_3, y_4) = (y_3^8.y_4^{32}).(1)$
Polynomial $P_1 = 0$
Polynomial $P_2 = 1$
Found non-degenerate set.
Vertices of the face of the non-degenerate set: $\{[0, 8.00],\ [8.00, 0]\}$.
Compact face F_4:
$g_{1_{|F4}} \circ \pi_{4,5}(y_4, y_5) = y_4^9.y_5 + y_4^9 = (y_4^9.y_5^0).(y_5 + 1)$
$g_{2_{|F4}} \circ \pi_{4,5}(y_4, y_5) = y_4^9.y_5 = (y_4^9.y_5^1).(1)$
Compact face F_4 is non-degenerate.
Vertices of the face of the non-degenerate set: $\{[0, 1.00],\ [9.00, 0]\}$.

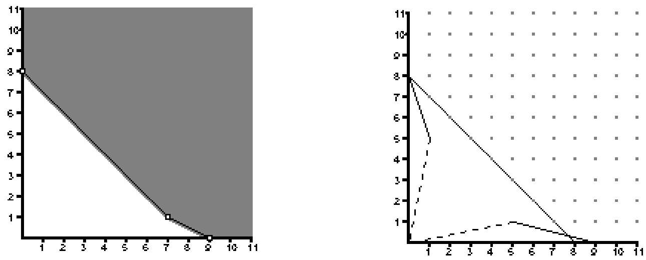


Figure 2: Polyhedron $C(\overline{I})$. $C(\overline{I})$ showed as intersection of the highest levels(lines) of each non-degenerate set

6. An example in $\mathbb{C}[x, y, z]$

Let $I = \langle g_1, g_2, g_3 \rangle$; with $g_1(x, y, z) = x^3 - y^3z^5 + y^6$, $g_2(x, y, z) = xyz$ and $g_3(x, y, z) = 3z^4 + x^3y$.

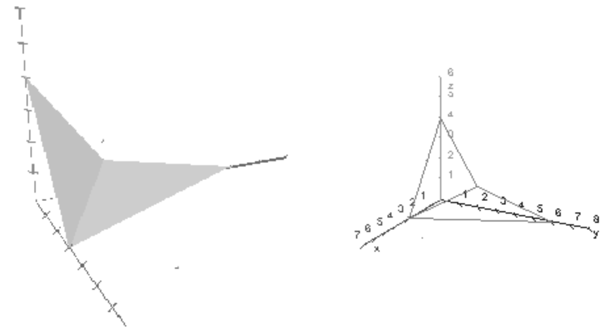


Figure 3: Newton polyhedron $\Gamma_+(I)$ Edges of $\Gamma_+(I)$

Report

Generators of the ideal:
$g_1(x,y,z) = X^3 - Y^2Z$, $g_2(x,y,z) = Y^6 + X^3YZ^3$, $g_3(x,y,z) = 3Z^4 + X^5Y^2$

Normal Vectors: $a^1 = (1,0,0)$, $a^2 = (8,9,6)$, $a^3 = (6,3,12)$, $a^4 = (0,1,0)$, $a^5 = (0,0,1)$

Compact faces of the Newton polyhedron:
$F_2 = \{[3,0,0][0,2,1][0,0,4]\}$, $F_3 = \{[3,0,0][0,6,0][0,2,1]\}$.

Functions of the toroidal embedding

$$h_{1,2,3}(y_1,y_2,y_3) = (y_1^1.y_2^8.y_3^6, y_1^0.y_2^9.y_3^3, y_1^0.y_2^6.y_3^{12})$$
$$h_{2,3,4}(y_2,y_3,y_4) = (y_2^8.y_3^6.y_4^0, y_2^9.y_3^3.y_4^1, y_2^6.y_3^{12}.y_4^0)$$
$$h_{3,4,5}(y_3,y_4,y_5) = (y_3^6.y_4^0.y_5^0, y_3^3.y_4^1.y_5^0, y_3^{12}.y_4^0.y_5^1)$$

Composition of the functions with the generators of the ideal:
Generator g_1:

$$g_1 \circ h_{1,2,3}(y_1,y_2,y_3) = y_1^3y_2^{24}y_3^{18} - y_2^{24}y_3^{18}$$
$$g_1 \circ h_{2,3,4}(y_2,y_3,y_4) = y_2^{24}y_3^{18} - y_2^{24}y_3^{18}y_4^2$$
$$g_1 \circ h_{3,4,5}(y_3,y_4,y_5) = y_3^{18} - y_3^{18}y_4^2y_5$$

Generator g_2:

$$g_2 \circ h_{1,2,3}(y_1,y_2,y_3) = y_2^{54}y_3^{18} + y_1^3y_2^{51}y_3^{57}$$
$$g_2 \circ h_{2,3,4}(y_2,y_3,y_4) = y_2^{54}y_3^{18}y_4^6 + y_2^{51}y_3^{57}y_4$$
$$g_2 \circ h_{3,4,5}(y_3,y_4,y_5) = y_3^{18}y_4^6 + y_3^{57}y_4y_5^3$$

Generator g_3:

$$g_3 \circ h_{1,2,3}(y_1,y_2,y_3) = 3y_2^{24}y_3^{48} + y_1^5y_2^{58}y_3^{36}$$
$$g_3 \circ h_{2,3,4}(y_2,y_3,y_4) = 3y_2^{24}y_3^{48} + y_2^{58}y_3^{36}y_4^2$$
$$g_3 \circ h_{3,4,5}(y_3,y_4,y_5) = 3y_3^{48}y_5^4 + y_3^{36}y_4^2$$

Restrictions of the generators to the compact faces:
Generator g_1: $g_{1|F_2}(X,Y,Z) = X^3 - Y^2Z$, $g_{1|F_3}(X,Y,Z) = X^3 - Y^2Z$.
Generator g_2: $g_{2|F_2}(X,Y,Z) = 0$, $g_{2|F_3}(X,Y,Z) = Y^6$.
Generator g_3: $g_{3|F_2}(X,Y,Z) = 3Z^4$, $g_{3|F_3}(X,Y,Z) = 0$.
Check of the non-degeneracy of the compact faces:
Compact face F_2:

$$g_{1|F_2} \circ h_{2,3,4}(y_2,y_3,y_4) = y_2^{24}y_3^{18} - y_2^{24}y_3^{18}y_4^2 = (y_2^{24}y_3^{18}).(1 - y_4^2)$$
$$g_{2|F_2} \circ h_{2,3,4}(y_2,y_3,y_4) = 0 = (1).(0)$$

$$g_{3|F_2} \circ h_{2,3,4}(y_2, y_3, y_4) = 3y_2^{24}y_3^{48} = (y_2^{24}y_3^{48}).(3)$$

$$g_{1|F_2}(X, Y, Z) = X^3 - Y^2Z = (1).(X^3 - Y^2Z)$$
$$g_{2|F_2}(X, Y, Z) = 0 = (1).(0)$$
$$g_{3|F_2}(X, Y, Z) = 3Z^4 = (1).(3Z^4)$$

Groebner basis for this restriction: $\{X^3 - Y^2Z, Z^4\}$
Compact face F_2 is degenerate
Searching for the non-degenerate set of this face

$$g_{1|F_2}(X, Y, Z) = 0$$
$$g_{2|F_2}(X, Y, Z) = X^3YZ^3$$
$$g_{3|F_2}(X, Y, Z) = 0$$

Groebner basis for this restriction: $\{Z^4, X^3 - Y^2Z\}$
Degenerate set of compact face F_2
Searching for the non-degenerate set of this face

$$g_{1|F_2}(X, Y, Z) = 0$$
$$g_{2|F_2}(X, Y, Z) = Y^6$$
$$g_{3|F_2}(X, Y, Z) = 0$$

Groebner basis for this restriction: $\{Y^6, Z^4, X^3 - Y^2Z\}$
Found non-degenerate set of compact face F_2
Compact face F_3:

$$g_{1|F_3} \circ h_{3,4,5}(y_3, y_4, y_5) = y_3^{18} - y_3^{18}y_4^2y_5 = (y_3^{18}).(1 - y_4^2y_5)$$
$$g_{2|F_3} \circ h_{3,4,5}(y_3, y_4, y_5) = y_3^{18}y_4^6 = (y_3^{18}y_4^6).(1)$$
$$g_{3|F_3} \circ h_{3,4,5}(y_3, y_4, y_5) = 0 = (1).(0)$$

$$g_{1|F_3}(X, Y, Z) = X^3 - Y^2Z = (1).(X^3 - Y^2Z)$$
$$g_{2|F_3}(X, Y, Z) = Y^6 = (1).(Y^6)$$
$$g_{3|F_3}(X, Y, Z) = 0 = (1).(0)$$

Groebner basis for this restriction: $\{X^3 - Y^2Z, Y^6\}$
Compact face is degenerate
Searching for the non-degenerate set of this face

$$g_{1|F_3}(X, Y, Z) = 0$$
$$g_{2|F_3}(X, Y, Z) = 0$$

$$g_{3|F_3}(X,Y,Z) = X^5Y^2$$

Groebner basis for this restriction: $\{X^2Y^4Z, Y^6, X^3 - Y^2Z\}$
Degenerate set of compact face F_3
Searching for the non-degenerate set of this face

$$g_{1|F_3}(X,Y,Z) = 0$$

$$g_{2|F_3}(X,Y,Z) = 0$$

$$g_{3|F_3}(X,Y,Z) = 3Z^4$$

Groebner basis for this restriction $\{X^2Y^4Z, Z^4, Y^6, X^3 - Y^2Z\}$
Found non-degenerate set of compact face F_3

Monomials $x^ay^bz^c$ in the Integral Closure satisfy:

$$6a + 3b + 12c > 48$$

$$8a + 9b + 6c > 54$$

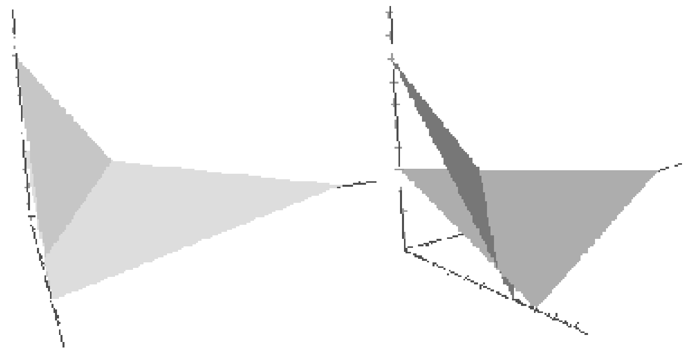


Figure 4: The polyhedron $C(\overline{I})$, showed as intersection of the highest levels (planes) of each non-degenerate set.

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