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Newton-Type Methods for Solution of the Electric Network Equations

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Abstract. Electric newtork equations give rise to interesting mathematical models that must be faced with efficient numerical optimization techniques. Here, the classical power flow problem represented by a nonlinear system of equations is solved by inexact Newton methods in which each step only approximately satisfies the linear Newton equation. The problem of restoring solution from the prior one comes when no real solution is possible. In this case, a constrained nonlinear least squares problem emerges and Newton and Gauss Newton methods for this formulation are employed.

1. Introduction

The development of methodologies for electric network equations has fundamental importance in power system analysis, yieding interesting formulations in terms of nonlinear optimization ([7],[3]).

The power flow problem consists in finding a solution for a system of nonlinear equations. One of the most widely used methods for solving this problem is the Newton's method with some form of factorization of the resulting Jacobian matrices, although the corresponding linear systems tend to be large and sparse ([5],[8]). Iterative techniques to the linear systems allow us to obtain approximate solutions to the Newton equations. This gives rise to the Inexact Newton methods for nonlinear systems.

When power systems become heavily loaded, there are situations where the power flow equations have no real solution. The associated problem here is restoring the solutions of the prior problem. This involves a Constrained Nonlinear Least Squares Problem (CNLSP) which is more complex than the load flow problem because the the linear systems are larger than the prior and ill conditioned. In this case, a combination of nonlinear least squares methods are needed ([4]).

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The objective of this work is to evaluate Newton type methods for electric network equations in these two formulations. These methods involve different ways of facing the Newton equations for both formulations.

In section 2, we present electric network equations, showing the equations, variables and technical data. Afterwards, in section 3, we describe and compare some preconditioning iterative methods for solving the nonsymmetric linear systems that come from the newtonian linearization of the first problem. In section 4, we present some numerical results of the second problem, solving the system derived from the CNLSP. Finally, in section 5 we give some concluding remarks.

2. Electric Network Equations

For each bus, say k, in an electric network there are four associated unknowns: tension node magnitude (V_k), tension node fase angle (θ_k), active power liquid injection (P_k) and reactive power liquid injection (Q_k). The power flow equations for nb bus systems (see [7]) are given by:

$$P_i^{cal} = G_{ii}V_i^2 + V_i \sum_{k\in\Omega_i} V_k[G_{ik}cos(\theta_i - \theta_k) + B_{ik}sen(\theta_i - \theta_k)],$$
$$Q_i^{cal} = -B_{ii}V_i^2 + V_i \sum_{k\in\Omega_i} V_k[G_{ik}sen(\theta_i - \theta_k) - B_{ik}cos(\theta_i - \theta_k)],$$

where $i = 1, \ldots, nb$ and Ω_k are the neighbour buses set at bus k. The terms G_{ik} and B_{ik} are the entries of the conductance and susceptance matrices respectively.

The Power Flow problem consists of finding the state (V, θ), such that

$$f(x) = \left[\begin{array}{c} (P^{law} - P^{cal}(V,\theta)) \\ (Q^{law} - Q^{cal}(V,\theta)) \end{array} \right] = 0,$$

where P^{law}, Q^{law} are the given lawsuits of the system. Therefore, one seeks the solution for a nonlinear system of algebraic equations.

3. Inexact Newton Methods

An alternative to the Newton's method for solving

$$f(x) = 0, \ f : \mathbb{R}^n \longrightarrow \mathbb{R}^n, \tag{3.1}$$

is to compute an intermediate solution p_k of the Newton equation,

$$J(x_k)p_k = -f(x_k), \tag{3.2}$$

where $J(x_k)$ is the Jacobian matrix at x_k. Computing the exact solution of (3.2) can be expensive if such matrix is large and sparse. Iterative linear algorithms give an approximate solution p_k giving rise to the Inexact Newton methods [8]. The iterate x_k is updated by $x_{k+1} = x_k + p_k$.

For terminating the iterative solver, the residual vector

$$r_k = J(x_k)p_k + f(x_k)$$

must satisfy

$$\|r_k\| \leq \eta_k \|f(x_k)\|, \qquad 0 \leq \eta_k < 1, \tag{3.3}$$

where η_k is the *forcing term*. The convergence can be made as fast as that of a sequence of the exact Newton iterations by taking η_k's to be sufficiently small [8].

Powerful iterative methods can be used to the Inexact Newton methods, and a good class of them are based on *Krylov Subspaces*.

3.1. Krylov Subspace Iterative Methods

Consider the linear system

$$Jp = b, \quad \mathrm{J} \in \mathbb{R}^{n \times n}, \tag{3.4}$$

which must be seen the same as (3.2): $J \equiv J(x_k)$, $p \equiv p_k$ and $b \equiv -f(x_k)$.

The Krylov Subspace is the following subspace of $\mathbb{R}^n$

$$K_m(J, v) = span\{v, Jv, J^2v, \dots, J^{m-1}v\}$$

for some vector $v \in \mathbb{R}^n$. The Krylov Subspace methods look for an approximate solution x^m in $K_m(J, r_0) \equiv K_m$., where $r^0 = b - Jx^0$. Different methods are associated with the projection-type approaches to find a suitable approximation x^m to the solution p^* of the system (3.4), which are:

(A) $b - Jx^m \perp K_m$;

(B) $\|b - Jx^m\|$ to be minimal over K_m;

(C) $b - Jx^m$ is orthogonal to some suitable $m - dimensional$ subspace;

(D) $\|x^* - x^m\|$ to be minimal over $J^T K_m(J^T, r^0)$.

The approaches (A) and (B) lead to the well known Generalized Minimum Residual (GMRES), that is based on the Arnoldi algorithm ([1],[6]). If in (C) we select the $k - dimensional$ subspace as $K_m(J^T, r^0)$, we obtain the Quasi-Minimum Residual (QMR) method, that is based on Lanczos Biorthogonalization algorithm ([2], [10]). Approach (D), which appeared more recently, gives rise to the transpose free methods, as Conjugate Gradient Squared (CGS) BiConjugate Gradient Stabilized (BiCG-Stab) methods ([10]). For more details see Saad [10], and a classical reference is Arnoldi [1].

Preconditioning techniques accelerates convergence of iterative methods. The motivation of preconditioning is to reduce the overall computational effort required to solve linear systems of equations by increasing the convergence rate of the underlying iterative algorithm. Assuming that the preconditioning matrix M is used in the left of the original system, this involves solving the preconditioned linear system

$$M^{-1}Jp = M^{-1}b \iff \tilde{J}p = \tilde{b}.$$

The choice of the preconditioner is crucial for the convergence process. It should be sparse, easy to invert and an apprroxiamtion as good as possible for J. In practice, M^{-1} is not computed directy, but only triangular systems involving the factors of M. In the particular method incomplete factorization, a lower triagular matrix L and an upper triangular matrix U are constructed that are good approximations of the LU factors of J, are also sparse. In this case, $M = LU$.

3.2. Numerical Results

The tests were accomplished using MATLAB 6 [9] with a Pentium 600 MHz. For solving the nonlinear system (3.1) we used the Newton Inexact method with stop criterion $\|f(x)\|_\infty < 10^{-3}$, for the nonlinear outer iterations.

The iterative solvers were accelerated by two Incomplete LU factorization (ILU) preconditioners. ILU(0), with zero level of fill scheme means that no fill-in is allowed out of the original data structure. In the ILU dropped scheme, the fill-in is discarded ever that $|L_{lm}|, |U_{lm}| \leq 0.01 \sum_{k=1}^{n} |\bar{J}_{km}|$, where $\bar{J}$ is the partial factored form of J.

Two electrical systems with 118 and 340 buses were tested. The first, results in a 201×201 sparse linear system and the other in a 626×626 system. In Figure 1 the frame of the systems with nonzero elements (nz) are shown. The picture represents the pattern of the original Jacobian matrix which becomes more complex as the dimension of the system increases.

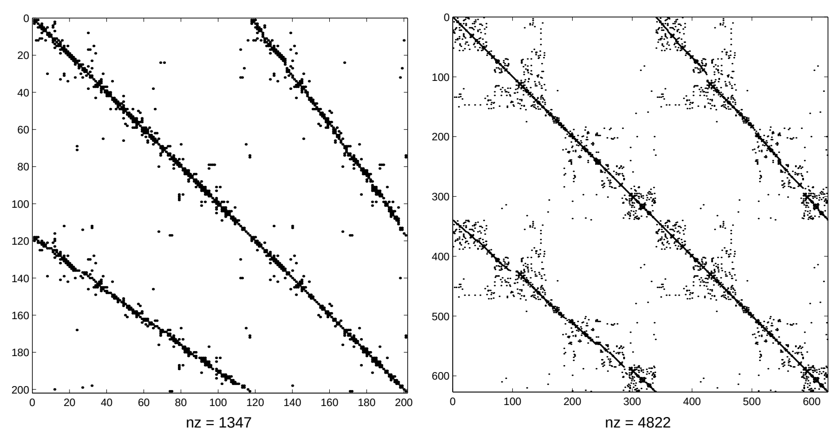

Figure 1: Matrices frame for 118 and 340 buses systems respectively

Results of the iterative algorithms are shown in Table 1. Each entry represents the amount of iteration at each Newton outer step. The CPU time was normalized for each bus system.

Table 1: Performs of the Methods

Method	Iteration Number at each Newton Step		Time(sec)	
	118 buses	340 buses	118 buses	340 buses
GMRES	1; 10; 14; 14	2; 6; 10; 12; 13; 14	1.13	1.18
CGS	1; 10; 8	1; 4; 9; 9; 11	1	1
BiCG-Stab	1; 7; 12	1; 2; 6; 6; 9;10	1	1.14
QMR	0; 4; 11; 13	2; 4; 10; 14; 13; 15; 17	1	1.12

The difference between the sequence of iterations illustrates the different nature of these methods. Some of them require more matrix-multiplication as well as preconditioning steps yielding more expensive iterations. Despite that, linear algorithm had similar behavior for both bus systems, specially for 118 buses. It can be observed that the preconditioner plays an important role, because in general few iterations were needed, specially in the beginning.

From the nonlinear point of view, the results correspond to $\eta_k = 0.06$ for 118 buses case and $\eta_k = 0.01$ for 340 buses case. These were the best choice for η_k and it means that high values of $\|f(x)\|$ dominated the criterion (3.7).

4. Nonlinear Least Square Solution

Sometimes, the nonlinear system (3.1) has no real solution due to the overload of the electric network. In these cases the following Nonlinear Least Squares Problem must be solved:

$$min\ F(x) = min\frac{1}{2}f(x)^T f(x) = min\ \frac{1}{2}\|f(x)\|_2^2,$$

In real problems, it is common that some buses must satisfy the lawsuit exigencies and the problem becomes the Constrained Nonlinear Least Squares Problem (CNLSP):

$$\begin{array}{rl} Min & \frac{1}{2}\|f(x)\|_2^2 \\ s.t. & h(x) = 0 \end{array}, \tag{4.1}$$

where the vectorial function $h(x)$ contains the unbalancing $(\Delta P, \Delta Q)$ for the buses within residual.

Problem (4.1) can be solved by obtaining Newton equations similar to (3.2). The basic idea is to iteratively solve several symmetric $n \times n$ linear systems

$$\nabla^2 \mathcal{L}(x_k, \lambda_k) p_k = -\nabla \mathcal{L}(x_k, \lambda_k), \tag{4.2}$$

where $\mathcal{L}(x,\lambda) = \frac{1}{2}f(x)^T f(x) - \lambda^T h(x)$. Then

$$\nabla^2 \mathcal{L}(x,\lambda) = \begin{bmatrix} J(x)^T J(x) + D(x,\lambda) & H^T(x) \\ H(x) & 0 \end{bmatrix},$$

where $D(x,\lambda) = \sum_{i=1}^{m} f_i(x)\nabla^2 f_i(x) + \sum_{i=1}^{k} \lambda_i \nabla^2 h_i(x)$, J and H are the Jacobian matrices of f and h respectively. For each step p_k, we find the next approximated solution by setting $x_{k+1} = x_k + \alpha_k p_k$, where α_k is the *line search parameter*, computed by solving

$$\alpha_k = \min_{\alpha} z(x_k + \alpha p_k). \tag{4.3}$$

The function $z : \mathbb{R}^n \longrightarrow \mathbb{R}$ is called the merit function. An alternative to the Newton method is the Gauss-Newton Method in which the second order information is dropped, which means $D(x,\lambda) \equiv 0$.

4.1. Numerical Results

System (4.2) is also sparse but iterative linear techniques are not quite satisfactory due to the ill conditioning of the system (see Table 2). Therefore, we use the ordinary Newton method making use of sparse symmetric factorization.

Table 2: Condition Number of both systems

	Condition Number*	
Iter	118 Buses (221 × 221)	340 buses (832×832)
1	5.0058×10^{6}	2.6208×10^{10}
2	1.7007×10^{7}	2.7096×10^{10}
3	2.8755×10^{6}	2.6300×10^{10}
4	1.0622×10^{7}	2.0251×10^{10}
5	1.2733×10^{7}	2.7374×10^{10}
6	1.1765×10^{7}	2.7796×10^{10}
7	1.2337×10^{7}	2.7927×10^{10}
8	1.2384×10^{7}	

*Approximate condition number $\approx k_2(A) = \|A\|_2 \|A^{-1}\|_2$

In nonlinear optimization, line search strategies can be used to improve the convergence of local methods. Here, we use one based on the augmented Lagrangian merit function,

$$z(x,\lambda) = \frac{1}{2}\|f(x)\|_2^2 + \lambda^T h(x) + \frac{1}{2\mu}\|h(x)\|_2^2, \tag{4.4}$$

where λ is the Lagrange multiplier vector and μ is the penalization parameter (see [8] for more details).

The best combination is shown in Table 3. Some initial Gauss Newton iterations with line search at the beginning followed by Newton iterations were the best results.

Newton's method, even with line search did not converge, for example, for 340 bus system. The penalization parameter μ was initilized with one for the merit function and updated iteratively by: $\mu_{j+1} \leftarrow 0.5\mu_j$ in (4.4) to obtain enough decrease in (4.3). About two ($j = 1, 2$) searches were necessary for each Gauss-Newton iteration and no search was necessary for Newton's method. The behavior of the Lagrangian gradients are shown in Table 3. We can note the linear behavior of the Lagrangian gradient norm in the sequence of iterations. In theory, this occurs with singular systems, and Table 2 confirms the ill conditioning of the system.

Table 3: Lagrangian Gradient behavior

	118 buses		340 buses	
Iter	$\|\nabla\mathcal{L}\|_\infty$	Iter. Type*	$\|\nabla\mathcal{L}\|_\infty$	Iter. Type*
1	2.4263×10^{2}	GN	1.5231×10^{4}	GN
2	7.5553×10^{1}	GN	2.1948×10^{3}	GN
3	2.4974	N	1.2519×10^{2}	GN
4	1.8953	N	4.6269	N
5	4.4772×10^{-1}	N	3.2312×10^{-1}	N
6	6.6687×10^{-2}	N	2.2665×10^{-2}	N
7	3.3115×10^{-3}	N	1.3766×10^{-4}	N
8	5.2351×10^{-6}	N		

* GN: Gauss Newton; N: Newton

5. Conclusions

Electric network equations can be efficiently solved by Newton-type Methods. The fundamental contribution of this work was to show that well conditioned power flow systems can be solved by Inexact Newton methods. When no real solution is available, the systems are ill conditioned and nonlinear least squares constrained methodologies can be employed adequately.

For the power flow problem, the systems required good approximation considering the criterion (3.3), and the preconditioner plays an important role. Despite the different number of iterations of the iterative methods, results were similar and good solutions were obtained. The problem of restoring the power flow solution is more complex and was satisfactorily solved by a combination of Gauss-Newton and Newton's method. The linear convergence rate of Newton's method confirms the ill conditioning of the systems. In high dimension, both problems become more complex and more numerical investigations are needed. Iterative algorithms are adequate for parallel matrix computation and ill conditioned systems must be faced with special methodologies.

Resumo. As equações da rede elétrica resultam em modelos matemáticos interessantes que devem ser abordados por técnicas eficientes de otimização numérica.

Neste trabalho, o problema clássico de fluxo de potência, representado por um sistema de equações não lineares, é resolvido utilizando métodos de Newton inexatos, os quais em cada passo satisfaz o sistema newtoniano apenas aproximadamente. O problema de restauração das equações da rede elétrica surge quando as equações de fluxo não apresentam solução real. Neste caso, um problema de quadrados mínimos com restrição aparece e usamos uma combinação dos métodos de Newton e Gauss Newton para esta formulação.

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