

An Explicit Jordan Decomposition of Companion Matrices¹

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Abstract. We derive a closed form for the Jordan decomposition of companion matrices including properties of generalized eigenvectors. As a consequence, we provide a formula for the inverse of confluent Vandermonde matrices and results on sensitivity of multiple roots of polynomials.

1. Introduction

We are concerned with the Jordan form of companion matrices of the form

$$C = [e_2, e_3, \dots, e_{m-1}, -a], \quad (1.1)$$

where e_i denotes the i th column of I , the $m \times m$ identity matrix, and $a^T = [a_0, a_1, \dots, a_{m-1}]$ with $a_i \in \mathbb{C}$ (superscript T denotes the transpose of a vector or matrix). Such a form plays an essential role in solving a variety of problems in science and engineering [1, 2, 3, 5, 9, 11]. Let $\lambda_1, \dots, \lambda_p$ denote the p distinct eigenvalues of C and let m_1, \dots, m_p denote their respective algebraic multiplicities. Then, since C is a nonderogatory matrix [8, 10], a particular Jordan decomposition of C can be written as

$$\begin{pmatrix} J_{\lambda_1} & & \\ & \ddots & \\ & & J_{\lambda_p} \end{pmatrix} = \begin{bmatrix} L_1 \\ \vdots \\ L_p \end{bmatrix} C [R_1 \dots R_p] \equiv L C R,$$

where for $i = 1, \dots, p$,

$$J_{\lambda_i} = \begin{pmatrix} \lambda_i & 1 & & \\ & \ddots & \ddots & \\ & & \lambda_i & 1 \\ & & & \lambda_i \end{pmatrix} \in \mathbb{C}^{m_i \times m_i}$$

and

$$L R = R L = I, \quad (1.2)$$

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where $R_i \in \mathbb{C}^{m \times m_i}$ and $L_i \in \mathbb{C}^{m_i \times m}$. The columns of R_i (resp. L_i^*) represent a right (resp. left) Jordan chain associated with λ_i the leading eigenvector being $R_i e_1^{[m_i]}$ (resp. $L_i^* e_{m_i}^{[m_i]}$). The star symbol denotes conjugate transpose, i.e., $L^* = \bar{L}^T$, and $e_j^{[m_i]}$ is the j -th column of the $m_i \times m_i$ identity matrix.

Companion matrix C has the property that the coefficients of its characteristic polynomial are the a 's themselves: $\pi(t) = \det(tI - C) = a_0 + a_1 t + \dots + a_{m-1} t^{m-1} + t^m$. Because of this, issues regarding roots of polynomials can be discussed using the Jordan decomposition of C . The goal of this work is to describe a closed form for the Jordan decomposition of C , concentrating on properties of generalized eigenvectors and issues regarding the sensitivity of the roots of π .

2. Explicit Jordan Decomposition

We start by providing a technical result.

Lemma 1. *For arbitrary $\lambda \in \mathbb{C}$ we set $\phi(\lambda) = [1, \lambda, \dots, \lambda^{m-1}]^T \in \mathbb{C}^m$ and define by $\phi^{(m)}(\lambda)$ the m -th derivative of $\phi(\lambda)$ with respect to λ . Let H be the $m \times m$ matrix*

$$H = \begin{bmatrix} a_1 & a_2 & \cdots & a_{m-1} & 1 \\ a_2 & \cdots & a_{m-1} & 1 & \\ \vdots & \vdots & 1 & & \\ a_{m-1} & 1 & & & \\ 1 & & & & \end{bmatrix}. \quad (2.1)$$

Then for any integers i and j , there holds $\frac{\phi^{(i)T}(\lambda)}{i!} H \frac{\phi^{(j)}(\lambda)}{j!} = \frac{\pi^{(i+j+1)}(\lambda)}{(i+j+1)!}$.

Proof. The proof is done by induction on i without difficulties. \square

Proposition 2. *Define $r_i = H \frac{\phi^{(i-1)}(\lambda_i)}{(i-1)!}$. The set of vectors $\{r_1, \dots, r_{m_i}\}$ is a right Jordan chain of C associated with the eigenvalue λ_i and r_1 is the leading right eigenvector. Similarly, define $\check{l}_i = \frac{\bar{\phi}^{(m_i-i)}(\lambda_i)}{(m_i-i)!}$. The set $\{\check{l}_1, \dots, \check{l}_{m_i}\}$ is a left Jordan chain of C associated with the eigenvalue λ_i and \check{l}_{m_i} is the leading left eigenvector. The left and right generalized Jordan chains are normalized so that*

$$\check{L}_i R_i \equiv \begin{bmatrix} \check{l}_1^* \\ \vdots \\ \check{l}_{m_i}^* \end{bmatrix} [r_1 \dots r_{m_i}] = \begin{pmatrix} \alpha_1 & \alpha_2 & \cdot & \alpha_{m_i-1} & \alpha_{m_i} \\ & \cdot & \cdot & \cdot & \alpha_{m_i-1} \\ & & \cdot & \cdot & \cdot \\ & & & \cdot & \alpha_2 \\ & & & & \alpha_1 \end{pmatrix} \equiv F_i, \quad (2.2)$$

where $\alpha_i = \frac{\pi^{(m_i+i-1)}(\lambda_i)}{(m_i+i-1)!}$.

Proof. For arbitrary λ of multiplicity q consider the vectors r_1, \dots, r_q . It is clear that these vectors are linearly independent. Thus, if we set $r_0 = 0$, we have to prove that r_1 is a right eigenvector of C associated with λ and that

$$(C - \lambda I)r_j = r_{j-1}, \quad 1 \leq j \leq q. \quad (2.3)$$

For this, if $x = [x_1 \cdots x_m]^T$ is a right eigenvector of C associated with λ then

$$Cx = \lambda x \iff \begin{cases} -a_0 x_m & = \lambda x_1 \\ x_1 - a_1 x_m & = \lambda x_2 \\ \vdots & \\ x_{m-1} - a_{m-1} x_m & = \lambda x_m \end{cases} .$$

This shows that x_m cannot vanish otherwise x would be the 0 vector. Setting $x_m = 1$ it is easy to see that $x = H\phi(\lambda)$ and thus one has

$$CH\phi(\lambda) = \lambda H\phi(\lambda). \tag{2.4}$$

We now prove conditions (2.3). Taking derivative with respect to λ in (2.4) we have

$$CH\phi^{(1)}(\lambda) = H\phi(\lambda) + \lambda H\phi^{(1)}(\lambda). \tag{2.5}$$

This shows that (2.3) holds for $j = 2$, and an inductive argument obtained by repeated differentiation of (2.5) concludes the proof in the case of the right generalized eigenvectors. A similar proof can be obtained for the generalized left eigenvectors by starting with $\phi(\lambda)^T C^T = \phi(\lambda)^T \lambda$ instead of (2.4) and taking the derivatives of this equality. The normalization factors α_i are a consequence of Lemma 1. \square

To obtain the Jordan decomposition, we transform the left Jordan chain so that the normalization (1.2) holds.

Proposition 3. Define $L_l^* = [l_1, \dots, l_{m_l}] = \check{L}_l^* F_l^{-*}$. The set $\{l_1, \dots, l_{m_l}\}$ is a left Jordan chain of C associated with the eigenvalue λ_l , l_{m_l} being the leading left eigenvector. The left and right generalized Jordan chains are normalized so that $L_l R_l = I \in \mathbb{R}^{m_l \times m_l}$. Similarly, if $\check{R}_l = [\check{r}_1, \dots, \check{r}_{m_l}] = [r_1, \dots, r_{m_l}] F_l^{-1}$, then $\{\check{r}_1, \dots, \check{r}_{m_l}\}$ is a right Jordan chain of C associated with λ_l , and $\check{L}_l \check{R}_l = I \in \mathbb{R}^{m_l \times m_l}$.

Proof. Let γ_i be defined by the recursion

$$\begin{aligned} \gamma_1 &= 1/\alpha_1, \\ \gamma_{i+1} &= -\frac{1}{\alpha_1} \sum_{k=1}^i \alpha_{i-k+2} \gamma_k, \quad i = 1, \dots, m_l - 1 \end{aligned} ,$$

in such a way that

$$G = \begin{pmatrix} \gamma_1 & \gamma_2 & \cdot & \gamma_{m_l-1} & \gamma_{m_l} \\ & \cdot & \cdot & \cdot & \gamma_{m_l-1} \\ & & \cdot & \cdot & \cdot \\ & & & \cdot & \gamma_2 \\ & & & & \gamma_1 \end{pmatrix} = F_l^{-1} .$$

The set $\{\check{l}_{m_l} \dots \check{l}_1\}$ forms a right Jordan chain of C^T associated with λ_l . For any nonsingular matrix X commuting with J_{λ_l} , $[\check{l}_{m_l} \dots \check{l}_1] X$ is a right Jordan chain C^T associated with λ_l . By definition of the l_i 's, $[l_{m_l} \dots l_1] = [\check{l}_{m_l} \dots \check{l}_1] \check{G}$. A direct computation shows that G commutes with J_{λ_l} , which implies that $\{\check{l}_{m_l}, \dots, \check{l}_1\}$ is a right Jordan chain of C^T associated with λ_l , that is, $\{l_1, \dots, l_{m_l}\}$

is a left Jordan chain of C associated with λ_l . Additionally, since by definition of the γ_i 's, $GF_l = F_lG = I$, it follows

$$\begin{bmatrix} l_1^* \\ \vdots \\ l_{m_l}^* \end{bmatrix} [r_1 \dots r_{m_l}] = G \begin{bmatrix} \check{l}_1^* \\ \vdots \\ \check{l}_{m_l}^* \end{bmatrix} [r_1 \dots r_{m_l}] = I,$$

and the first part of the proposition is proved. The proof of the remaining part is a consequence of Eq. (2.2) since $[r_1 \dots r_{m_l}]F_l^{-1}$ is a right Jordan chain of C associated with λ_l as we have seen that F^{-1} commutes with J_{λ_l} . \square

An immediate consequence of Prop. 3 is an explicit formula for computing the inverse of confluent Vandermonde matrices as described below

Corollary 4 (Inversion formula). *Let \check{L} be the confluent Vandermonde matrix defined by $\check{L}^* = [\check{L}_1^* \dots \check{L}_p^*]$. Then*

$$\check{L}^{-1} = [R_1 \dots R_p]F^{-1} \quad \text{with} \quad F = \text{diag}(F_1, \dots, F_p).$$

2.1. Numerical illustration: Jordan decomposition

We present an illustration of the above notions for $m = 5$, $(\lambda_1, m_1) = (1, 2)$, $(\lambda_2, m_2) = (2, 2)$, $(\lambda_3, m_3) = (3, 1)$, in which case, $\pi(t) = (t-1)^2(t-2)^2(t-3)$. We show how to obtain easily a Jordan form of the companion matrix associated with π . Note that $\pi(t) = t^5 - 9t^4 + 31t^3 - 51t^2 + 40t - 12$.

- Case of $\lambda = 1$, $m_1 = 2$.

From $(t-2)^2(t-3) = t^3 - 7t^2 + 16t - 12$ and $(t-1)(t-2)^2(t-3) = t^4 - 8t^3 + 23t^2 - 28t + 12$ follows using Prop. 5 and the definition of \check{l}_i that

$$R_1 = \begin{pmatrix} 12 & -12 \\ -28 & 16 \\ 23 & -7 \\ -8 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad \check{L}_1^* = \begin{pmatrix} 0 & 1 \\ 1 & 1 \\ 2 & 1 \\ 3 & 1 \\ 4 & 1 \end{pmatrix}.$$

From $\pi^{(2)}(1)/2 = -2$ and $\pi^{(3)}(1)/6 = 5$, we obtain

$$F_1 = \begin{pmatrix} -2 & 5 \\ 0 & -2 \end{pmatrix}, \quad F_1^{-1} = \frac{1}{4} \begin{pmatrix} -2 & -5 \\ 0 & -2 \end{pmatrix}, \quad \text{and} \quad L_1^* = \check{L}_1^* F_1^{-1} = \frac{1}{4} \begin{pmatrix} -5 & -2 \\ -7 & -2 \\ -9 & -2 \\ -11 & -2 \\ -13 & -2 \end{pmatrix}.$$

- The same calculation for the two remaining roots gives

$$R = [R_1, R_2, R_3] = \begin{pmatrix} 12 & -12 & 6 & -3 & 4 \\ -28 & 16 & -17 & 7 & -12 \\ 23 & -7 & 17 & -5 & 13 \\ -8 & 1 & -7 & 1 & -6 \\ 1 & 0 & 1 & 0 & 1 \end{pmatrix}$$

and

$$L^* = [L_1^*, L_2^*, L_3^*] = \frac{1}{4} \begin{pmatrix} -5 & -2 & 4 & -4 & 1 \\ -7 & -2 & 4 & -8 & 3 \\ -9 & -2 & 0 & -16 & 9 \\ -11 & -2 & -16 & -32 & 27 \\ -13 & -2 & -64 & -64 & 81 \end{pmatrix},$$

yielding a Jordan decomposition $RJL = C$, where J (a Jordan matrix) and C are of the form

$$J = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 3 \end{pmatrix}, \quad C = \begin{pmatrix} 0 & 0 & 0 & 0 & 12 \\ 1 & 0 & 0 & 0 & -40 \\ 0 & 1 & 0 & 0 & 51 \\ 0 & 0 & 1 & 0 & -31 \\ 0 & 0 & 0 & 1 & 9 \end{pmatrix}.$$

3. Generalized Eigenvector Properties

It is known that right eigenvectors of companion matrices like C in (1.1) can be computed by finding the vector of coefficients of $\pi(t)/(t - \lambda_l)$ (see, e.g., Toh and Trefethen [6] or Bezerra and Bazán [3, Prop. 2.4]). In our notation this reads

$$\pi_1(t) = \pi(t)/(t - \lambda_l) \equiv \phi(t)^T r_1.$$

In what follows we extend this result to the complete right Jordan chain.

Proposition 5. Define $\pi_i(t) = \phi(t)^T r_i$ ($i = 1, \dots, m_l$) where r_i are generalized right eigenvectors of C as introduced in Prop. 2. Then π_i is a monic polynomial of degree $m - i$ of the form

$$\pi_i(t) = (t - \lambda_l)^{m_l - i} \prod_{\substack{j=1 \\ j \neq l}}^p (t - \lambda_j)^{m_j}. \tag{3.1}$$

Proof. It is clear that all π_i are monic polynomials of degree $m - i$. The definition of the r_i 's and successive differentiation imply

$$\begin{aligned} \pi_1(t) &= \phi^T(t) H \phi(\lambda_l) \\ \pi_1^{(1)}(t) &= \phi^{(1)T}(t) H \phi(\lambda_l) \\ &\vdots \\ \pi_1^{(i)}(t) &= \phi^{(i)T}(t) H \phi(\lambda_l) \\ &\vdots \\ \pi_1^{(m_l - 1)}(t) &= \phi^{(m_l - 1)T}(t) H \phi(\lambda_l). \end{aligned} \tag{3.2}$$

If $t = \lambda_l$, Prop. 1 implies that for $i = 1, \dots, m_l$, (3.2) becomes

$$\pi_i(\lambda_l) = \pi_1^{(i-1)}(\lambda_l) = (i - 1)! \phi^T(\lambda_l) H \frac{\phi^{(i-1)}(\lambda_l)}{(i - 1)!} = (i - 1)! \pi^{i-1}(\lambda_l).$$

But since λ_l is a multiple root of π , this equality implies that λ_l is a root of π_i ($i = 1 \dots m_l - 1$) and a recursive argument shows that this root is of multiplicity $m_l - i$. If $t = \lambda_k \neq \lambda_l$, a similar procedure and the existing biorthogonality condition between left and right generalized eigenvectors leads to

$$\pi_i(\lambda_k) = \pi_1^{(i-1)}(\lambda_k) = (i-1)! \phi^T(\lambda_l) H \frac{\phi^{(i-1)}(\lambda_k)}{(i-1)!} = 0,$$

which concludes the proof. \square

It is instructive to describe the meaning of this proposition. Let C_i ($i = 1, \dots, m_l - 1$) denote the $(m-i) \times (m-i)$ companion matrix associated with the polynomial π_i and, for $i = 1, \dots, m_l$, let \check{r}_i be the vector formed by taking the first $m-i+1$ components of r_i . Then, with the convention that $C_0 = C$, the proposition ensures that

$$C_{i-1} \check{r}_i = \lambda_l \check{r}_i, \quad i = 1, \dots, m_l, \quad (3.3)$$

and λ_l is a simple eigenvalue of the companion matrix C_{m_l-1} . For future reference, the left eigenvector of C_{m_l-1} , will be denoted by $\psi(\lambda_l)$. It is defined by

$$\psi(\lambda_l) = [1, \bar{\lambda}_l, \dots, \bar{\lambda}_l^{m-m_l}]^T. \quad (3.4)$$

4. Condition Estimation

We analyze the sensitivity of the roots of $\pi(t)$ to perturbations in the coefficients a_j viewing the roots as eigenvalues of the companion matrix C . Let $\tilde{\pi}(t)$ denote the monic polynomial with coefficients $\tilde{a}_i = a_i + \Delta a_i$ and let \tilde{C} denote the associated companion matrix. Then, depending on the way the perturbations \tilde{a}_j are measured, different condition numbers for λ can be obtained. Suppose for instance that the Δa_j 's are assumed to satisfy the componentwise inequalities

$$|\Delta a_i| \leq \epsilon \alpha_i, \quad i = 0, \dots, m-1, \quad (4.1)$$

where α_j are arbitrary non negative real numbers, and let $\tilde{\lambda}_j$, $j = 1, \dots, d$, denote the eigenvalues of \tilde{C} for ϵ small enough. For the so-called *componentwise* model of perturbations defined by (4.1), we have the definition below, where for simplicity, λ_l and its corresponding multiplicity m_l will be denoted by λ and d , respectively, and $|\Delta \lambda| = \max_{j=1, \dots, d} |\lambda - \tilde{\lambda}_j|$.

Definition 6. [4] *The componentwise relative condition number of the root λ of multiplicity d is defined by*

$$\kappa^C(\lambda) = \lim_{\epsilon \rightarrow 0} \sup_{|\Delta a_j| \leq \epsilon \alpha_j} \frac{|\Delta \lambda|}{|\lambda| \epsilon^{1/d}}. \quad (4.2)$$

A precise description of $\kappa^C(\lambda)$ is given in the proposition below.

Proposition 7. *Suppose the perturbations Δa_j satisfy (4.1). Then the componentwise relative condition number of the root λ of multiplicity d , $\kappa^C(\lambda)$, is*

$$\kappa^C(\lambda) = \frac{1}{|\lambda|} \left(\frac{d! \sum_{j=0}^{m-1} |\lambda^j| \alpha_j}{|\pi^{(d)}(\lambda)|} \right)^{1/d}. \tag{4.3}$$

Proof. A sketch of the proof is as follows. Let $\{\tilde{\lambda}, \tilde{r}\}$ be a right eigenpair of $\tilde{C} = C + \Delta C$. If $\tilde{\lambda} = \lambda + \Delta\lambda$, $\tilde{r} = r + \Delta r$, then $(C + \Delta C)(r + \Delta r) = (\lambda + \Delta\lambda)(r + \Delta r)$, iff $C\Delta r + \Delta C r + \Delta C \Delta r = \lambda \Delta r + \Delta \lambda r + \Delta \lambda \Delta r$. Now since by Prop. 2.2, the right eigenvector \tilde{r} of \tilde{C} satisfies $r + \Delta r = \tilde{H}\phi(\tilde{\lambda})$, where $\tilde{H} = H + \Delta H$ has the same structure as H but with entries $\tilde{a}_j = a_j + \Delta a_j$, it can be proved that the m -th component of Δr equals zero. From this and the fact that $\Delta C = -\Delta a e_m^{[m]*}$ where $\Delta a = [\Delta a_0, \dots, \Delta a_{m-1}]^T$ gives $\Delta C \Delta r = 0$. Some algebraic manipulations lead then to the following first order result

$$\Delta \lambda^d = \frac{d! \phi^T(\lambda) \Delta a}{\pi^{(d)}(\lambda)}, \tag{4.4}$$

from which the proof follows. □

Remark. Another condition number for λ can be readily obtained when the perturbations are assumed to satisfy

$$\|\Delta a\|_2 \leq \delta \alpha, \tag{4.5}$$

where α is an arbitrary positive real number (e.g., $\alpha = \|a\|_2$). This gives rise to the so-called *normwise relative condition number* $\kappa(\lambda)$ which is given by

$$\kappa(\lambda) = \frac{1}{|\lambda|} \left(\frac{d! \|\phi(\lambda)\| \|a\|}{|\pi^{(d)}(\lambda)|} \right)^{1/d}. \tag{4.6}$$

When the perturbations are measured in a normwise absolute sense, i.e., when $\alpha = 1$ in (4.5), the *normwise absolute condition* $\kappa_a(\lambda)$ can be shown to satisfy

$$\kappa_a(\lambda) = |\lambda| \kappa(\lambda). \tag{4.7}$$

Observe that the multiple eigenvalue λ can be viewed as a simple eigenvalue of the companion matrix C_{d-1} in (3.3) (see the remark after Prop. 5). In what follows we shall see that $\kappa_a(\lambda)$ and $\kappa(\lambda)$ can be related to the Wilkinson condition number of λ viewed as a simple eigenvalue of C_{d-1} ; this condition number is defined by [12]

$$\kappa_w(\lambda) = \frac{\|\psi(\lambda)\|_2 \|\check{r}_d\|_2}{|\psi^*(\lambda) \check{r}_d|}. \tag{4.8}$$

Proposition 8. *Assume that the perturbation Δa satisfies the model (4.5) with $\alpha = \|a\|_2$. Then the condition number $\kappa(\lambda)$ given in (4.6) satisfies*

$$\kappa(\lambda) \leq \frac{1 + |\lambda|}{|\lambda|} \left(\frac{\|a\|_2^2}{1 + \|a\|_2^2} \right)^{\frac{1}{2d}} \kappa_w(\lambda)^{1/d} \left(\frac{\|\phi(\lambda)\|_2}{\|\psi(\lambda)\|_2} \right)^{1/d}. \tag{4.9}$$

Proof. Observe that using Prop. 2 and the property $\phi^T(\lambda)r_d = \psi^*(\lambda)\check{r}_d$ (see (3.3) again), (4.6) can be rewritten as

$$\kappa(\lambda) = \frac{1}{|\lambda|} \left(\frac{\|\phi(\lambda)\|_2 \|r_d\|_2}{|\phi^T(\lambda)r_d|} \right)^{1/d} \left(\frac{\|a\|_2}{\|r_d\|_2} \right)^{1/d} = \frac{1}{|\lambda|} \kappa_w(\lambda)^{1/d} \frac{\|\phi(\lambda)\|_2^{1/d} \|a\|_2^{1/d}}{\|\psi(\lambda)\|_2^{1/d} \|r_d\|_2^{1/d}}.$$

We now recall that vector r_d in (4.9) contains the coefficients of the monic polynomial π_d that is of degree $m - d$ (see the remark after Prop. 5). Set $q = m - d - 1$, $r_d = [c_0, c_1, \dots, c_q, 1, 0 \dots 0]^T \equiv [\check{r}_d^T, 0 \dots 0]^T$ and collect the coefficients of the polynomial $(t - \lambda)^d = b_0 + b_1 t + \dots + b_{d-1} t^{d-1} + t^d$ in a vector $b = [b_0, b_1, \dots, b_{d-1}, 1]^T$. Some algebraic manipulations and Prop. 5 give then

$$\pi(t) = (t - \lambda)^d \pi_d(t) \Leftrightarrow \begin{bmatrix} a \\ 1 \end{bmatrix} = B \check{r}_d. \tag{4.10}$$

where B is an $(m + 1) \times (m - d + 1)$ lower triangular Toeplitz matrix whose first column vector is $[b_0, b_1, \dots, b_{d-1}, 1, 0, \dots, 0]^T$. Since $\|B\|_2 \leq \sqrt{\|B\|_1 \|B\|_\infty} = \|B\|_1$, (4.10) gives

$$1 + \|a\|_2^2 \leq \|B\|_2^2 \|r_d\|_2^2 = \|B\|_1^2 \|r_d\|_2^2.$$

Hence, taking into account that $\|B\|_1 = (1 + |\lambda|)^d$, as the b_i 's are of the form $b_i = \lambda^{d-i} C_i^d$ ($i = 0, \dots, d$) where C_i^d are binomial coefficients of the expansion $(t - \lambda)^d$, we have $\|r_d\|_2 \geq \sqrt{1 + \|a\|_2^2} / (1 + |\lambda|)^d$. The proof follows on replacing this lower bound in (4.9). \square

An immediate conclusion is that if d is not large and $|\lambda|$ is a moderate number then $\kappa(\lambda)$ essentially depends on $\kappa_w(\lambda)$. Thus if λ is a well conditioned eigenvalue of the deflated companion matrix C_{d-1} (or equivalently, a well conditioned simple root of $\pi_{d-1}(t)$) and the ratio $\|\phi(\lambda)\|_2 / \|\psi(\lambda)\|_2$ is rather small, then moderate values for $\kappa(\lambda)$ may be expected. We notice however that even if $\kappa(\lambda)$ is small, the error in λ strongly depends on the multiplicity d and the size of the perturbations Δa_j . In general, if the perturbations Δa_j are small enough, the relative error in λ can be estimated by the rule

$$|\Delta \lambda| / |\lambda| \approx \kappa(\lambda) \delta^{1/d}. \tag{4.11}$$

4.1. Numerical illustration: Condition estimation

We consider the polynomial $\pi(t) = (t - \lambda)^5(1 + t + \dots + t^{15})$ with $\lambda = (1 + 9s) + si$, $0 \leq s \leq 2$. This example is designed to illustrate the role of the deflated polynomial π_{d-1} (in this case $d = 5$, see Prop. 5) in estimating the sensitivity of a multiple root. In fact, as in in this case the deflated polynomial $\pi_{d-1}(t) = (t - \lambda)(1 + t + \dots + t^{15})$ reduces to the polynomial $t^{16} - 1$ when $\lambda = 1$, all roots of which are known to be extremely well-conditioned [7, Example 4.3], small condition numbers for the multiple root λ can be expected provided that $\lambda \approx 1$, the conditioning being more favorable for the (simple) roots of $\pi(t)$ (the roots of $1 + t + \dots + t^{15}$). Indeed, if the simple roots of $\pi(t)$ are denoted by $\check{\lambda}_k$, it can be proved that

$$\kappa_a(\lambda) = \frac{(1 + |\lambda|^2 + |\lambda|^4 + \dots + |\lambda|^{38})^{0.1}}{\prod_{k=1}^{15} |\lambda - \check{\lambda}_k|^{0.2}}, \quad \text{and} \quad \kappa_a(\check{\lambda}_k) = \frac{\sqrt{5} |\check{\lambda}_k - 1|}{8 |\check{\lambda}_k - \lambda|^5}.$$

λ	$\kappa(\lambda)$	$\kappa_a(\lambda)$	κ_w	ρ	$ \Delta\lambda / \lambda $
$19 + 2i$	$1.3169e + 1$	$1.0480e + 1$	$3.7970e + 0$	$1.3322e + 5$	$1.3169e - 1$
$15 + 1.5i$	$1.0724e + 1$	$8.6469e + 0$	$3.7443e + 0$	$5.1642e + 4$	$1.0724e - 1$
$10 + i$	$7.4747e + 0$	$6.2102e + 0$	$3.6221e + 0$	$1.0201e + 4$	$7.4747e - 2$
$5 + 0.5i$	$3.9220e + 0$	$3.4955e + 0$	$3.2743e + 0$	$6.3756e + 2$	$3.9220e - 2$
$1.45 + 0.05i$	$1.5800e + 0$	$1.1384e + 0$	$1.7266e + 0$	$4.4310e + 0$	$1.5800e - 2$
1	$1.2693e + 0$	$7.7495e - 1$	$1.0000e + 0$	$1.1180e + 0$	$1.2693e - 2$

Table 1: Condition numbers, ratio $\rho = \|\phi(\lambda)\|_2/\|\psi(\lambda)\|_2$ and predicted errors .

Some numerical results displayed in Table 1 corresponding to several λ 's confirm the theoretical prediction. The table includes condition numbers, the predicted eigenvalue error described in (4.11), and the ratio $\rho = \|\phi(\lambda)\|_2/\|\psi(\lambda)\|_2$. Also, and mainly to verify the theoretical prediction of the error, approximate roots obtained from polynomials with coefficients $\tilde{a}_j = a_j + \Delta a_j$ where Δa_j are random numbers satisfying a normwise relative error $\delta = 10^{-10}$, are displayed in Figure 1. All computations were performed using MATLAB.

The results confirm that moderate values of $\kappa(\lambda)$ do not necessarily imply small eigenvalue errors when the multiplicity is rather large and that reasonably small errors can be expected when both the Wilkinson condition number $\kappa_w(\lambda)$ and the ratio ρ are small. The relative insensitivity of simple roots is also apparent, as predicted.

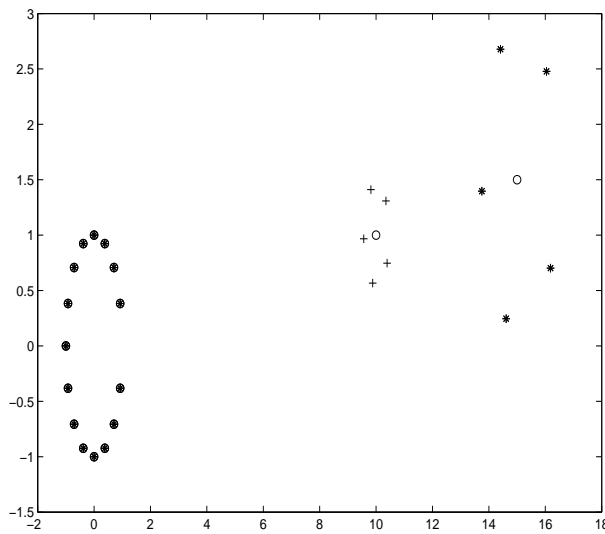


Figure 1: Exact and approximate eigenvalues: Case 1: $\lambda = 15 + 1.5i$. \circ : Exact eigenvalue, $*$: Approximate eigenvalue. Case 2: $\lambda = 10 + i$. \circ : Exact eigenvalue, $+$: Approximate eigenvalue.

Final Remark

We have presented an explicit Jordan decomposition of companion matrices and results on sensitivity of roots of polynomials. Efficient implementation of the formula for the inverse of confluent Vandermonde matrices is the subject of ongoing work. The results on sensitivity of multiple roots, on the other hand, contribute to clarify an important problem in numerical analysis usually overlooked so far.

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