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Properties of Fuzzy Implications obtained via the Interval Constructor

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Abstract. This work considers an interval extension of fuzzy implication based on the best interval representation of continuous t-norms. Some related properties can be naturally extended and that extension preserves the behaviors of the implications in the interval endpoints.

1. Introduction

Fuzzy logic is a new form of information theory that is related to but independent of interval mathematics. However, when intervals can be thought as a particular type of fuzzy set, or when interval degrees of membership are used to deal with the uncertainties of the belief degrees of a specialist, it is natural and interesting to consider an interval fuzzy approach [8, 9, 16, 17]. Based on this approach, in this work we consider the comparative analysis started with the work of Bedregal and Takahashi [5] integrating both areas: (1) fuzzy logic, as a formal mathematical theory for the representation of uncertainty concerned with fuzzy set theory, which is crucial for the management and control of real systems; and (2) interval mathematics, providing the correctness criteria and the optimality properties of numerical computations, and offering a more reliable modelling of real systems.

Emerging from the fuzzy set theory, a fuzzy logic can be understood as a superset of classical logic to handle the concept of partial truth and considers the extension principle, from a crisp (discrete) to a continuous (fuzzy) form. This means that truth values are distributed as degrees between completely true and completely false, both represented by the endpoints 0 and 1 of the real unity interval $[0, 1]$. In addition, continuous t-norms are modelled as standard truth functions of conjunction, and residue as standard truth functions of implication. Thus, the extension of classical logic connectives to the real unit interval is fundamental for the studies on fuzzy logic and, therefore, it is essential for the development of fuzzy systems. This extension

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must preserve the behaviors of the connectives at the interval endpoints, i.e., for the crisp values. Moreover, it has been a consensus in this research area that, for the case of the connectives conjunction and disjunction, this extension must also preserve other important properties, such as commutative and associative properties, which result in the notions of triangular norms and triangular conorms.

Fuzzy implications play an important role in fuzzy logic, both in the broad sense (heavily applied to fuzzy control, analysis of vagueness in natural language and techniques of soft-computing) and in the narrow sense (developed as a branch of many-valued logic which are able to investigate deep logical questions). However, there is not a consensus among researchers which extra properties fuzzy implications must satisfy. In the literature, several fuzzy implication properties have already been proved and their interrelationship with the other kinds of connectives were presented. Recently, Bedregal and Takahashi [5, 6], working towards the connection between fuzzy logic and interval mathematics, have provided interval extensions for the fuzzy connectives considering both correctness (accuracy) and optimality aspects, as properly shown in [20]. In a categorical approach, the interval generalization related to both t-norms and automorphisms can be seen as interval representation satisfying the correctness principle of interval computations. In this work, for the case of fuzzy implication, the authors only considered the properties proposed by Fodor and Roubens and the class of R-implications [10].

In this present work, the interval constructor introduced by Bedregal and Takahashi in [5, 6] is used in order to show that most of the properties considered nowadays for fuzzy implication in the literature are also preserved. Then, we focus attention in the interval extension of fuzzy t-norm and fuzzy negation in Sections 3 and 4, respectively. Further analysis of properties met by interval fuzzy implications are focused in Section 4. Finally, our main results are summarized in Section 5.

2. Best Interval Representations

Consider the real unit interval $U = [0, 1]$. Let $\mathbb{U}$ be the set of subintervals of U, i.e., $\mathbb{U} = \{[a, b] \mid 0 \leq a \leq b \leq 1\}$. The interval set has two projections $l : \mathbb{U} \longrightarrow U$ and $r : \mathbb{U} \longrightarrow U$ defined by $l([a, b]) = a$ and $r([a, b]) = b$. As a convention, for each $X \in \mathbb{U}$ the projections $l(X)$ and $r(X)$ will also be denoted by $\underline{X}$ and $\overline{X}$, respectively.

Several natural partial orders can be defined on $\mathbb{U}$ [4]. The most used orders in the context of interval mathematics and considered in this work, are the following.

1. *Product*: $X \leq Y$ if and only if $\underline{X} \leq \underline{Y}$ and $\overline{X} \leq \overline{Y}$,
2. *Inclusion order*: $X \subseteq Y$ if and only if $\underline{X} \geq \underline{Y}$ and $\overline{X} \leq \overline{Y}$.

A function $F : \mathbb{U}^n \longrightarrow \mathbb{U}$ is an *interval representation* of a function $f : U^n \longrightarrow U$ if, for each $\vec{X} \in \mathbb{U}^n$ and $\vec{x} \in \vec{X}$, $f(\vec{x}) \in F(\vec{X})$ [20]. An interval can be seen as a representation of a subset of real numbers. In this case we can say that an interval X is a better representation of a real r than an interval Y if X is narrower than Y, i.e., if $X \subseteq Y$. Trivially, this notion could be extended for tuples on intervals.

Analogously, an interval function $F : \mathbb{U}^n \longrightarrow \mathbb{U}$ is a *better representation* of the function $f : U^n \longrightarrow U$ than $G : \mathbb{U}^n \longrightarrow \mathbb{U}$, denoted by $G \sqsubseteq F$, if for each $\vec{X} \in \mathbb{U}^n$, $F(\vec{X}) \subseteq G(\vec{X})$. For each function $f : U^n \longrightarrow U$, $\widehat{f} : \mathbb{U}^n \longrightarrow \mathbb{U}$ defined by

$$\widehat{f}(\vec{X}) = [\inf\{f(\vec{x}) \mid \vec{x} \in \vec{X}\}, \sup\{f(\vec{x}) \mid \vec{x} \in \vec{X}\}]$$

is well defined and for any other interval representation F of f, $F \sqsubseteq \widehat{f}$. In other words, $\widehat{f}$ returns a narrower interval than any other interval representation of f and is therefore its *best interval representation* [20]. Thus, $\widehat{f}$ has the *optimality property* of interval algorithms mentioned by Hickey et al. [12], when seen as an algorithm to compute f. Notice that if f is continuous in the usual sense, then for each $\vec{X} \in \mathbb{U}^n$, $\widehat{f}(\vec{X}) = \{f(\vec{x}) \mid \vec{x} \in \vec{X}\} = f(\vec{X})$.

In the interval analysis, the continuity of interval functions can be obtained as an extension of the real ones.[5] Moore and Scott continuities are the two most common continuity notions used in interval mathematics. The former is concerned with the metric distance $d(X, Y) = max(\mid \overline{X} - \overline{Y} \mid, \mid \underline{X} - \underline{Y} \mid)$ defined over the space of Moore intervals, with intervals seen as a subspace of the real plane emphasizing the related notion of proximity. In the latter, the set of real intervals with reverse inclusion order can be defined as a continuous domain (consistently complete continuous dcpo) [1] whose objects are intervals interpreting partial information of real numbers. Let $E = (E, \leq_E)$ and $D = (D, \leq_D)$ be directed complete partially ordered set (dcpo's). A function $f : E \to D$ is called Scott continuous if it is monotonic (i.e., $x \leq_E y \Rightarrow f(x) \leq_D f(y)$) and preserves the least upper bound (supremum) of directed sets (i.e., for each directed set $A \subseteq D$, $f(\bigsqcup X) = \bigsqcup f(X)$). The main result in [20] (p. 240) involving Scott and Moore continuities is the following:

Theorem 2.1. *Let $f : \Re \longrightarrow \Re$ be a real function. f is continuous iff $\widehat{f}$ is Scott continuous iff $\widehat{f}$ is Moore continuous.*

Clearly, the theorem 2.1 can be adapted to our context, i.e., for U^n instead of $\Re$.

3. Interval T-Norms

Given a t-norm based propositional fuzzy calculus, one can construct the corresponding predicate calculus, which is axiomatizable w.r.t. the general algebraic semantics [11]. A triangular norm, **t-norm** for short, is a function $T : U^2 \to U$ that is commutative, associative, monotonic and has 1 as neutral element. Following the approach introduced in [5], an extension of the t-norm notion for $\mathbb{U}$ is considered:

Definition 3.1. *A function $\mathbb{T} : \mathbb{U}^2 \to \mathbb{U}$ is an **interval t-norm** if it is commutative, associative, monotonic w.r.t. the product and inclusion order and $[1, 1]$ is a neutral element.*

Proposition 3.1. *If T is a t-norm then $\widehat{T} : \mathbb{U}^2 \to \mathbb{U}$ is an interval t-norm.*

Proof. See [5]. □

[5] A survey relating continuity notions can be found in [19].

4. Interval Fuzzy Negation

Recall that a function $N : U \to U$ is a *fuzzy negation* if

N1: $N(0) = 1$ and $N(1) = 0$.
N2: If $x \geq y$ then $N(x) \leq N(y)$, $\forall x, y \in U$.

Fuzzy negations satisfying the involutive property:
N3: $N(N(x)) = x$, $\forall x \in U$,
are called *strong fuzzy negations* [14, 7].

Definition 4.1. *An interval function* $\mathbb{N} : \mathbb{U} \longrightarrow \mathbb{U}$ *is an* interval fuzzy negation *if, for any* X, Y *in* $\mathbb{U}$, *the following properties hold*

$\mathbb{N}$*1:* $\mathbb{N}([0,0]) = [1,1]$ *and* $\mathbb{N}([1,1]) = [0,0]$.
$\mathbb{N}$*2: If* $X \geq Y$ *then* $\mathbb{N}(X) \leq \mathbb{N}(Y)$.
$\mathbb{N}$*3: If* $X \subseteq Y$ *then* $\mathbb{N}(X) \supseteq \mathbb{N}(Y)$.

If $\mathbb{N}$ *is also meets the involutive property, it is a* strong *interval fuzzy negation:*
$\mathbb{N}$*4:* $\mathbb{N}(\mathbb{N}(X)) = X$, $\forall X \in \mathbb{U}$.

Theorem 4.1. *Let* $N : U \longrightarrow U$ *be a fuzzy negation. Then* $\widehat{N}$ *is an interval fuzzy negation. If* N *is a strong fuzzy negation then* $\widehat{N}$ *is a strong interval fuzzy negation.*

Proof. $\mathbb{N}$1: Trivially, $\mathbb{N}$1 is satisfied.

$\mathbb{N}$2: If $X \geq Y$ then $\overline{Y} \leq \overline{X}$ and $\underline{Y} \leq \underline{X}$ therefore, by N2 property, $\widehat{N}(X) = [N(\overline{X}), N(\underline{X})] \leq [N(\overline{Y}), N(\underline{Y})] = \widehat{N}(Y)$.

$\mathbb{N}$3: If $X \subseteq Y$ then $\overline{X} \leq \overline{Y}$ and $\underline{Y} \leq \underline{X}$ therefore, by N2 property, $\widehat{N}(X) = [N(\overline{X}), N(\underline{X})] \subseteq [N(\overline{Y}), N(\underline{Y})] = \widehat{N}(Y)$.

$\mathbb{N}$4: If $\mathbb{N}$ is strong, $\widehat{N}(\widehat{N}(X)) = \widehat{N}([N(\overline{X}), N(\underline{X})]) = [N(N(\underline{X})), N(N(\overline{X}))] = X$.

□

5. Fuzzy Implications

Several definitions for fuzzy implication have been given, see for example [2, 3, 7, 11, 10, 13, 15, 18, 21, 22, 23]. The unique consensus in these definitions is that the fuzzy implication should have the same behavior as the classical implication for the crisp case. Thus, $I : U^2 \longrightarrow U$ is a *fuzzy implication* if

$$I(1,1) = I(0,1) = I(0,0) = 1 \text{ and } I(1,0) = 0.$$

In the following several reasonable extra properties that can be required for fuzzy implications are listed. In fact, each one of these properties can be found in most of the following papers: [2, 7, 11, 10, 13, 15, 18, 21, 22, 23].

I1: If $x \leq z$ then $I(x, y) \geq I(z, y)$.

I2: If $y \leq z$ then $I(x, y) \leq I(x, z)$.

I3: $I(0, y) = 1$, (falsity principle).

I4: $I(x, 1) = 1$, (neutrality principle).

I5: $I(x, I(y, z)) = I(y, I(x, z))$, (exchange principle).

I6: If $x \leq y$ then $I(x, y) = 1$, (boundary condition).

I7: $I(x, x) = 1$, (identity property).

I8: $I(x, y) \geq y$.

I9: I is a continuous function, (continuity condition).

I10: $I(x, y) = I(x, I(x, y))$.

Other two properties related to fuzzy implications with strong negation [7].

I11: If N is a strong negation, $I(x, y) = I(N(y), N(x))$, (contrapositive w.r.t.N).

I12: Let $N : U \longrightarrow U$. If $N(x) = I(x, 0)$ then N is a strong fuzzy negation.

Proposition 5.1. *Let I be a fuzzy implication. If I satisfies* I11 *and* I12 *then for each $x, y \in U$, $I(x, y) = I(I(y, 0), I(x, 0))$.*

Proof. See [2]. □

The law of importation relates some fuzzy implications with some t-norms [3]:

I13: Let T be a t-norm, $I(T(x, y), z) = I(x, I(y, z))$.

5.1. Interval Fuzzy Implications

Since real values in interval mathematics are identified with degenerate intervals, the minimal properties of fuzzy implications can be naturally extended for interval fuzzy degrees, considering the respective degenerate intervals. Thus, a function $\mathbb{I} : \mathbb{U}^2 \longrightarrow \mathbb{U}$ is a *fuzzy interval implication* if

$$\mathbb{I}([1,1],[1,1]) = \mathbb{I}([0,0],[0,0]) = \mathbb{I}([0,0],[1,1]) = [1,1] \text{ and } \mathbb{I}([1,1],[0,0]) = [0,0].$$

Notice that, by having two natural partial orders on $\mathbb{U}$ and two continuity notions, some extra properties can have two natural versions.

Extra properties of interval fuzzy implications

$\mathbb{I}$1: If $X \leq Z$ then $\mathbb{I}(X, Y) \geq \mathbb{I}(Z, Y)$.

$\mathbb{I}$2: If $Y \leq Z$ then $\mathbb{I}(X, Y) \leq \mathbb{I}(X, Z)$.

$\mathbb{I}$3: $\mathbb{I}([0, 0], Y) = [1, 1]$.

$\mathbb{I}$4: $\mathbb{I}(X, [1, 1]) = [1, 1]$.

$\mathbb{I}5$: $\mathbb{I}(X, \mathbb{I}(Y, Z)) = \mathbb{I}(Y, \mathbb{I}(X, Z))$.

$\mathbb{I}6$a: If $X \leq Y$ then $1 \in \mathbb{I}(X, Y)$.

$\mathbb{I}6$b: If $X \subseteq Y$ then $1 \in \mathbb{I}(X, Y)$.

$\mathbb{I}6$c: If $[x, x] \leq Y$ then $\mathbb{I}([x, x], Y) = [1, 1]$.

$\mathbb{I}6$d: If $X \leq [y, y]$ then $\mathbb{I}(X, Y[y, y]) = [1, 1]$.

$\mathbb{I}7$: $1 \in \mathbb{I}(X, X)$.

$\mathbb{I}8$: $\mathbb{I}(X, Y) \geq Y$.

$\mathbb{I}9$a: $\mathbb{I}$ is a Moore continuous function.

$\mathbb{I}9$b: $\mathbb{I}$ is a Scott continuous function.

$\mathbb{I}10$a: $\mathbb{I}(X, Y) \subseteq \mathbb{I}(X, \mathbb{I}(X, Y))$.

$\mathbb{I}10$b: $\mathbb{I}([x, x], Y) = \mathbb{I}([x, x], \mathbb{I}([x, x], Y))$.

$\mathbb{I}11$: Let $\mathbb{N}$ be a strong fuzzy negation. If $\mathbb{I}(X, Y) = \mathbb{I}(\mathbb{N}(Y), \mathbb{N}(X))$ then $\mathbb{I}$ is contrapositive w.r.t. $\mathbb{N}$.

$\mathbb{I}12$: If $\mathbb{N} : \mathbb{U} \longrightarrow \mathbb{U}$, $\mathbb{N}(X) = \mathbb{I}(X, [0, 0])$ then $\mathbb{N}$ is a strong interval fuzzy negation.

$\mathbb{I}13$: Let $\mathbb{T}$ be an interval t-norm. $\mathbb{I}(\mathbb{T}(X, Y), Z) = \mathbb{I}(X, \mathbb{I}(Y, Z))$.

Starting from any fuzzy implication it is always possible to obtain canonically an interval fuzzy implication. Then, the interval fuzzy implication also meets the optimality property and preserves the same properties satisfied by the fuzzy implication. In the next two propositions, the best interval representation of fuzzy implication is shown to be an inclusion-monotonic function in both arguments, and the proofs are straightforward, following from the definition of $\widehat{I}$.

Proposition 5.2. *If I is a fuzzy implication then $\widehat{I}$ is an interval fuzzy implication.*

Proposition 5.3. *Let I be a fuzzy implication. For each $X_1, X_2, Y_1, Y_2 \in \mathbb{U}$, if $X_1 \subseteq X_2$ and $Y_1 \subseteq Y_2$ then $\widehat{I}(X_1, Y_1) \subseteq \widehat{I}(X_2, Y_2)$.*

Theorem 5.1. *Let I be a fuzzy implication. If I satisfies a property* Ik*, for some* $\mathrm{k} = 1, \ldots, 10$ *then $\widehat{I}$ satisfies the property* $\mathbb{I}$k.

Proof. $\mathbb{I}1$: If $u \in \widehat{I}(X, Y)$, then there exist $x \in X$ and $y \in Y$ such that $I(x, y) = u$. If $X \leq Z$, then there exists $z \in Z$ such that $x \leq z$. So, by I1, it holds that $u = I(x, y) \geq I(z, y)$. On the other hand, if $v \in \widehat{I}(Z, Y)$, then there exist $z \in Z$ and $y \in Y$ such that $I(z, y) = v$. If $X \leq Z$ then $x \leq z$ for some $x \in X$. So, by I1, it holds that $I(x, y) \geq I(z, y) = v$. Therefore, for each $u \in \widehat{I}(X, Y)$, there is $v \in \widehat{I}(Z, Y)$ such that $u \geq v$, and, for each $v \in \widehat{I}(Z, Y)$, there is $u \in \widehat{I}(X, Y)$ such that $u \geq v$. Hence, it holds that $\widehat{I}(X, Y) \geq \widehat{I}(Z, Y)$.

$\mathbb{I}2$: If $u \in \widehat{I}(X,Y)$, then there exist $x \in X$ and $y \in Y$ such that $I(x,y) = u$. If $Y \leq Z$, then there exists $z \in Z$ such that $y \leq z$. So, by I2, it holds that $u = I(x,y) \leq I(x,z)$. On the other hand, if $v \in \widehat{I}(X,Z)$, then there exist $z \in Z$ and $x \in X$ such that $I(x,z) = v$. If $Y \leq Z$ then $y \leq z$, for some $y \in Y$. So, by I2, it holds that $I(x,y) \geq I(x,z) = v$. Thus, for each $u \in \widehat{I}(X,Y)$, there is $v \in \widehat{I}(X,Z)$ such that $u \leq v$, and for each $v \in \widehat{I}(X,Z)$, there is $u \in \widehat{I}(X,Y)$ such that $u \leq v$. Then, it holds that $\widehat{I}(X,Y) \leq \widehat{I}(X,Z)$.

$\mathbb{I}3$: Trivially, by I3, for each $y \in Y$, $I(0,y) = 1$, and so $\{I(0,y) : y \in Y\} = [1,1]$. Thus, since $\widehat{I}([0,0],Y)$ is the narrowest interval containing $\{I(0,y) : y \in Y\}$, then $\widehat{I}([0,0],Y) = [1,1]$.

$\mathbb{I}4$: Trivially, by I4, for each $x \in X$, $I(x,1) = 1$ and so $\{I(x,1) : x \in X\} = [1,1]$. Thus, since $\widehat{I}(X,[1,1])$ is the narrowest interval containing $\{I(x,1) : x \in X\}$, then $\widehat{I}(X,[1,1]) = [1,1]$.

$\mathbb{I}5$: If $u \in \widehat{I}(X,\widehat{I}(Y,Z))$, then there exist $x \in X$, $y \in Y$ and $z \in Z$ such that $I(x,I(y,z)) = u$. But, by I5, one has that $u = I(y,I(x,z))$. So, $u \in \widehat{I}(Y,\widehat{I}(X,Z))$, and, therefore, $\widehat{I}(X,\widehat{I}(Y,Z)) \subseteq \widehat{I}(Y,\widehat{I}(X,Z))$. Analogously, if $u \in \widehat{I}(Y,\widehat{I}(X,Z))$, then there exist $x \in X$, $y \in Y$ and $z \in Z$ such that $I(y,I(yx,z)) = u$. But, by I5, onde has that $u = I(x,I(y,z))$. So, $u \in \widehat{I}(X,\widehat{I}(XY,Z))$, and, therefore, $\widehat{I}(Y,\widehat{I}(X,Z)) \subseteq \widehat{I}(X,\widehat{I}(Y,Z))$. Hence, it holds that $\widehat{I}(X,\widehat{I}(Y,Z)) = \widehat{I}(Y,\widehat{I}(X,Z))$.

$\mathbb{I}6a$: If $X \leq Y$, then there exist $x \in X$ and $y \in Y$ such that $x \leq y$, and so, by I6, $I(x,y) = 1$. Therefore, it holds that $1 \in \widehat{I}(X,Y)$.

$\mathbb{I}6b$: If $X \subseteq Y$, then there exist $x \in X$ and $y \in Y$ such that $x \leq y$, and so, by I6, $I(x,y) = 1$. Therefore, it holds that $1 \in \widehat{I}(X,Y)$.

$\mathbb{I}6c$: If $[x,x] \leq Y$, then, for each $y \in Y$, $x \leq y$. So, by I6, for each $y \in Y$, $I(x,y) = 1$ and, therefore, it holds that $\widehat{I}([x,x],Y) = [1,1]$.

$\mathbb{I}6d$: If $X \leq [y,y]$, then for each $x \in X$, $x \leq y$. So, by I6, for each $x \in X$, $I(x,y) = 1$ and, therefore, it holds that $\widehat{I}(X,[y,y]) = [1,1]$.

$\mathbb{I}7$: If $x \in X$,then $I(x,x) = 1$, and so $1 \in \widehat{I}(X,X)$.

$\mathbb{I}8$: By I8, for each $x \in X$ and $y \in Y$, $I(x,y) \geq y$. So, $\widehat{I}(X,Y) \geq Y$.

$\mathbb{I}9a$ and $\mathbb{I}9b$: it is straightforward, following from Theorem 2.1.

$\mathbb{I}10a$: If $u \in \widehat{I}(X,Y)$, then there exist $x \in X$ and $y \in Y$ such that $I(x,y) = u$. So, by I10, $u = I(x,I(x,y))$, and, therefore, $u \in \widehat{I}(X,\widehat{I}(X,Y))$. Hence, it holds that $\widehat{I}(X,Y) \subseteq \widehat{I}(X,\widehat{I}(X,Y))$.

$\mathbb{I}10b$: By $\mathbb{I}$10a, $\widehat{I}([x,x],Y) \subseteq \widehat{I}([x,x],\widehat{I}([x,x],Y))$. So, it only remains to prove that $\widehat{I}([x,x],Y) \supseteq \widehat{I}([x,x],\widehat{I}([x,x],Y))$. Conisdering $u \in \widehat{I}([x,x],\widehat{I}([x,x],Y))$, then there exists $y \in Y$ such that $u = I(x,I(x,y))$. But, by I10, onde has that $I(x,I(x,y)) = I(x,y)$. So, $u \in \widehat{I}([x,x],Y)$, and, therefore, it holds that $\widehat{I}([x,x],Y) \supseteq \widehat{I}([x,x],\widehat{I}([x,x],Y))$.

□

The preservation of properties $\mathbb{I}11$, $\mathbb{I}12$ and $\mathbb{I}13$ will be proved separately, because another connective will be considered.

Proposition 5.4. *Let I be a fuzzy implication and N be a fuzzy strong negation, such that I is contrapositive w.r.t. N, i.e., it satisfies* I11. *Then $\widehat{I}$ is contrapositive w.r.t. $\widehat{N}$, i.e., it satisfies* $\mathbb{I}11$.

Proof. If $u \in \widehat{I}(X,Y)$, then there exist $x \in X$ and $y \in Y$ such that $I(x,y) = u$. But, by I11, one has that $I(x,y) = I(N(y), N(x))$. Since $N(y) \in \widehat{N}(Y)$ and $N(x) \in \widehat{N}(X)$, then $u \in \widehat{I}(\widehat{N}(Y), \widehat{N}(X))$. So, it holds that $\widehat{I}(X,Y) \subseteq \widehat{I}(\widehat{N}(Y), \widehat{N}(X))$. On the other hand, if $u \in \widehat{I}(\widehat{N}(Y), \widehat{N}(X))$, then there exist $v \in \widehat{N}(Y)$ and $w \in \widehat{N}(X)$ such that $I(v,w) = u$. But, since $v \in \widehat{N}(Y)$ and $w \in \widehat{N}(X)$, there exist $y \in Y$ and $x \in X$ such that $N(y) = v$ and $N(x) = w$. So, it holds that $I(N(y), N(x)) = u$. But, by I11, onde has that $I(N(y), N(x)) = I(x,y)$. Then, it holds that $u \in \widehat{I}(X,Y)$ and $\widehat{I}(X,Y) = \widehat{I}(\widehat{N}(Y), \widehat{N}(X))$. □

Proposition 5.5. *Let I be a fuzzy implication. If I satisfies a property* I12, *then the interval function $\mathbb{N} : \mathbb{U} \longrightarrow \mathbb{U}$, defined by $\mathbb{N}(X) = \widehat{I}(X, [0,0])$ is a strong interval fuzzy negation, i.e. satisfy the property* $\mathbb{I}12$.

Proof. By I12, $N : U \longrightarrow U$ defined by $N(x) = I(x,0)$ is a strong fuzzy implication, and, therefore, by Theorem 4.1, $\widehat{N}$ is a strong interval fuzzy negation. We will prove that $\mathbb{N} = \widehat{N}$. Consider $X \in \mathbb{U}$. If $u \in \mathbb{N}(X)$, then there exists $x \in X$ such that $I(x,0) = u$, and, therefore, such that $N(x) = u$. So, $u \in \widehat{N}(X)$. Conversely, if $u \in \widehat{N}(X)$ then there exist $x \in X$ such that $N(x) = u$. But, by I12, onde has that $I(x,0) = u$. So, it holds that $u \in \widehat{I}(X, [0,0])$, i.e., $u \in \mathbb{N}(X)$. Therefore, one concludes that $\mathbb{N} = \widehat{N}$. □

Proposition 5.6. *Let I be a fuzzy implication and T be a t-norm, such that I satisfy the law of importation w.r.t. T, i.e., it satisfies* I13. *Then $\widehat{I}$ satisfy the property* $\mathbb{I}13$ *w.r.t.* $\widehat{T}$.

Proof. If $u \in \widehat{I}(\widehat{T}(X,Y), Z)$, then there exist $v \in \widehat{T}(X,Y)$ and $z \in Z$ such that $u = I(v,z)$. But, if $v \in \widehat{T}(X,Y)$, then there exist $x \in X$ and $y \in Y$ such that $v = T(x,y)$. So, $u = I(T(x,y),z)$, and ,therefore, by property I13, $u = I(x, I(y,z))$. Thus, since $x \in X$ and $I(y,z) \in \widehat{I}(Y,Z)$, one has that $u \in \widehat{I}(X, \widehat{I}(Y,Z))$. Therefore, it holds that $\widehat{I}(\widehat{(T)}(X,Y), Z) \subseteq \widehat{I}(X, \widehat{I}(Y,Z))$. On the other hand, if $u \in \widehat{I}(X, \widehat{I}(Y,Z))$, then there exist $x \in X$ and $v \in \widehat{I}(Y,Z)$ such that $u = I(x,v)$. But, if $v \in \widehat{I}(Y,Z)$, then there exist $y \in Y$ and $z \in Z$ such that $v = I(y,z)$. So, $u = I(x, I(y,z))$, and, therefore, by property I13, onde has that $u = I(T(x,y),z)$. Thus, since $T(x,y) \in \widehat{T}(X,Y)$ and $z \in Z$, it holds that $u \in \widehat{I}(\widehat{T}(X,Y), Z)$. Therefore, one concludes that $\widehat{I}(\widehat{T}(X,Y), Z) = \widehat{I}(X, \widehat{I}(Y,Z))$. □

6. Final Remarks

In this paper, we mainly discussed under which conditions generalized fuzzy implications applied to interval values preserve properties of canonical forms generated by interval t-norms. It was shown that properties of fuzzy logic can be naturally extended for interval fuzzy degrees considering the respective degenerate intervals. The results are important not only for analyzing deductive systems in mathematical depth but also as foundations of methods of fuzzy logic in broad sense.

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Resumo. Este trabalho considera a extensão intervalar da implicação fuzzy baseada no conceito de melhor representação intervalar de t-normas contínuas, previamente introduzido por Bedregal e Takahashi. As correspondentes propriedades foram analisadas e verificou-se que o comportamento das implicações nos extremos do intervalo unitário pode ser preservado e naturalmente estendido.

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