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Some Results in Stability Analysis of Hybrid Dynamical Systems

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Abstract. In this paper we introduced a general model for the Hybrid Dynamical Systems and for such systems we introduced the usual concept of Lyapunov stability. Furthermore, we established two Principal Lyapunov Theorems and a converse theorem.

Keywords. Hybrid dynamical systems, Lyapunov stability, discontinuous dynamical systems.

1. Introduction

Hybrid systems are capable of exhibiting simultaneously several kinds of dynamic behavior, such that continuous-time dynamics, discrete-time dynamics, logic commands, and so forth.
At the present time, there does not appear to exist a satisfactory general model for hybrid dynamical systems which is situable for the qualitative analysis of such systems.
In the present paper we give a definition of hybrid dynamical system which covers a very large number of classes of hybrid systems and which is suitable for the qualitative analysis of such systems.

2. Hybrid Systems

2.1. Hybrid dynamical systems

Definition 2.1. *Given the set* X, $\prec \subset X \times X$ *it is an order relationship in* X *if for any* $x, y, z \in X$ *such that* $(x, y) \in \prec \Leftrightarrow x \prec y$ *:*
(i) $x \prec x$*;*
(ii) if $x \prec y$ *and* $y \prec x$ *then* $x = y$*;*
(iii) if $x \prec y$ *and* $y \prec z$ *then* $x \prec z$*.*

Definition 2.2. *We say that a function* $\phi \in C[[0, r], \mathbb{R}^+]$ *(respectively* $\phi \in C[\mathbb{R}^+, \mathbb{R}^+]$*) belongs to class* K *(*$\phi \in K$*), if* $\phi(0) = 0$ *and if* ϕ *is strictly increasing on* $[0, r]$ *(respectively on* $\mathbb{R}^+$*).*

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Furthermore, we say that a function $\phi \in K$ defined on $\mathbb{R}^+$ belongs to class $K\mathbb{R}$ if $\lim_{r\to\infty}\phi(r) = +\infty$.

Definition 2.3. *A metric space (T, ρ) is called a time space if:*
(i) T is completely ordered with order $\prec$;
(ii) T has a minimal element $t_{min} \in T$, that is, for any $t \in T$ and $t \neq t_{min}$, it is true that $t_{min} \prec t$;
(iii) for any $t_1, t_2, t_3 \in T$ such that $t_1 \prec t_2 \prec t_3$, it is true that $\rho(t_1, t_3) = \rho(t_1, t_2) + \rho(t_2, t_3)$;
(iv) T is unbounded from above, that is, for any $M > 0$, there exists a $t \in T$ such that $\rho(t, t_{min}) > M$.

Definition 2.4. *Let (X, d) be a metric space and let $A \subset X$. Let (T, ρ) be a time space, and let $T_0 \subset T$. For any fixed $a \in A$, $t_0 \in T_0$, we call a mapping $p(., a, t_0) : T_{a,t_0} \to X$ a motion if $p(t_0, a, t_0) = a$, where $T_{a,t_0} = \{t \in T : t_0 \preceq t\}$.*

Thus, we define hybrid dynamical systems:

Definition 2.5. *Let S be a set of motions, that is, $S \subset \{p(., a, t_0) \in \Lambda : p(t_0, a, t_0) = a\}$, where $\Lambda = \cup_{(a,t_0)\in(A\times T_0)}\{T_{a,t_0} \to X\}$. The five-tuple $\{T, X, A, S, T_0\}$ is called a hybrid dynamical system.*

2.2. Some qualitative characterizations

Definition 2.6. *Let $\{T, X, A, S, T_0\}$ be a hybrid dynamical system. A set $M \subset A$ is said to be invariant with respect to system S (that is, (S, M) is invariant) if $a \in M$ implies that $p(t, a, t_0) \in M$ for all $t \in T_{a,t_0}$ and all $t_0 \in T_0$ such that $p(., a, t_0) \in S$.*

Definition 2.7. *We call $x_0 \in A$ an equilibrium of a hybrid dynamical system $\{T, X, A, S, T_0\}$ if $(S, \{x_0\})$ is invariant.*

Definition 2.8. *(Lyapunov Stability) Let $\{T, X, A, S, T_0\}$ be a hybrid dynamical system and let $M \subset A$ be an invariant set of S.*
(i) We say that (S, M) is stable if for every $\epsilon > 0$, and $t_0 \in T_0$ there exists a $\delta = \delta(\epsilon, t_0) > 0$ such that $d(p(t, a, t_0), M) < \epsilon$ for all $t \in T_{a,t_0}$ and for all $p(., a, t_0) \in S$, whenever $d(a, M) < \delta$.
(ii) We say that (S, M) is uniformly stable if $\delta = \delta(\epsilon)$.
(iii)If (S, M) is stable and if for any $t_0 \in T_0$, there exists an $\eta = \eta(t_0) > 0$ such that, for every $\epsilon > 0$, there exists a $t_\epsilon \in T$ such that $d(p(t, a, t_0), M) < \epsilon$ whenever $t \in T$ and $t_\epsilon \preceq t$, for all $p(., a, t_0) \in S$ whenever $d(a, M) < \eta$, then (S, M) is called asymptotically stable.
(iv) We call (S, M) uniformly asymptotically stable if (S, M) is uniformly stable and if there exists a $\delta > 0$ and for every $\epsilon > 0$ there exists a $\tau = \tau(\epsilon) > 0$ such that $d(p(t, a, t_0), M) < \epsilon$ for all $t \in \{t \in T_{a,t_0} : \rho(t, t_0) \geq \tau\}$, and all $p(., a, t_0) \in S$ whenever $d(a, M) < \delta$.
(v) (S, M) is said to be exponentially stable if there exists $\alpha > 0$ such that for every $\epsilon > 0$ and $t_0 \in T_0$, there exists a $\delta = \delta(\epsilon) > 0$ such that $d(p(t, a, t_0), M) < \epsilon e^{-\alpha\rho(t,t_0)}$ for all $t \in T_{a,t_0}$ and for all $p(., a, t_0) \in S$, whenever $d(a, M) < \delta$.

Observation : The exponential stability of (S, M) implies the uniform asymptotic stability of (S, M).

Indeed, if (S, M) is exponentially stable then for every $\epsilon > 0$ and $t_0 \in T_0$, there exists $\alpha > 0$ and $\delta = \delta(\epsilon) > 0$ such that $d(p(t, a, t_0), M) < \epsilon e^{-\alpha\rho(t,t_0)}$ for all $t \in T_{a,t_0}$ and for all $p(., a, t_0) \in S$, whenever $d(a, M) < \delta$, therefore $d(p(t, a, t_0), M) < \epsilon e^{-\alpha\rho(t,t_0)} \leq \epsilon$ and (S, M) is uniformly stable.
Furthermore, for all $t \in T_{a,t_0}$ it is had $d(p(t, a, t_0), M) < \epsilon e^{-\alpha\rho(t,t_0)}$, thus, for any $\tau = \tau(\epsilon) > 0$ such that $t \in \{t \in T_{a,t_0} : \rho(t, t_0) \geq \tau\}$, $d(p(t, a, t_0), M) < \epsilon e^{-\alpha\rho(t,t_0)} < \epsilon$ and then (S, M) is uniformly asymptotically stable.

2.3. Embedding of hybrid dynamical systems into dynamical systems defined on $\mathbb{R}^+$

Any time space T can be embedded into the real space $\mathbb{R}^+$ by means of a mapping $g : T \to \mathbb{R}^+$, having the following properties:
(i) $g(t_{min}) = 0$, where t_{min} denotes the minimum element in T;
(ii) $g(t) = \rho(t, t_{min})$ for $t \neq t_{min}$.

If we let $R_1 = g(T)$, then g is an isometric mapping from T to R_1. Indeed, given $r \in R_1$, there exists $t \in T$ such that $r = g(t)$ and if $g(t_1) = g(t_2)$, for $t_1 \prec t_2$, since $\rho(t_{min}, t_2) = \rho(t_{min}, t_1) + \rho(t_1, t_2)$ it follows that $g(t_2) - g(t_1) = \rho(t_1, t_2)$ and then $t_1 = t_2$. Thus, g is a bijection from T onto R_1, furthermore, for $t_1 \prec t_2$ it is had $d(g(t_1), g(t_2)) = |g(t_2) - g(t_1)| = |\rho(t_1, t_2)| = \rho(t_1, t_2)$, therefore g is an isometric mapping from T to $R_1 = g(T)$.

Definition 2.9. *Let $\{T, X, A, S, T_0\}$ be a hybrid dynamical system, let $x \in A$ be fixed and let $g : T \to \mathbb{R}^+$ be the embedding mapping defined previously . Suppose that $p(., a, t_0) \in S$ is a motion defined on T_{a,t_0}. Let $\tilde{p}(., a, r_0) : \mathbb{R}^+_{r_0} \to X$, where $\mathbb{R}^+_{r_0} = \{r \in \mathbb{R}^+ : r \geq r_0\}$, be a function having the following properties:*
(i) *$r_0 = g(t_0)$;*
(ii) *$\tilde{p}(r, a, r_0) = p(g^{-1}(r), a, t_0)$ if $r \in R_1 = g(T)$;*
(iii) *$\tilde{p}(r, a, r_0) = x$ if $r \notin R_1 = g(T)$.*
We call $\tilde{p}(., a, r_0)$ the embedding of $p(., a, t_0)$ from T to $\mathbb{R}^+$ with respect to x.

Definition 2.10. *Let $\{T, X, A, S, T_0\}$ be a hybrid dynamical system and let $x \in A$. The hybrid dynamical system $\{\mathbb{R}^+, X, A, \tilde{S}, \mathbb{R}^+_0\}$ is called the embedding of $\{T, X, A, S, T_0\}$ from T to $\mathbb{R}^+$ with respect to x, where $\mathbb{R}^+_0 = g(T_0)$ and $\tilde{S}$ is the set of all $\tilde{p}(., a_0, r_0)$, such that $\tilde{p}(., a_0, r_0)$ is the embedding of $p(., a_0, t_0)$ with respect to x and $p(., a_0, t_0) \in S$.*

In view of the previous definitions, any hybrid dynamical system defined on an abstract time space T can be embedded into another hybrid dynamical system defined on real time space $\mathbb{R}^+$.
Furthermore, it should be noted that the various stability definitions given in subsection 2.2 for general hybrid dynamical systems $\{T, X, A, S, T_0\}$ with invariant set $M \subset A$ translate in a natural manner to the case of dynamical systems
$\{\mathbb{R}^+, X, A, \tilde{S}, \mathbb{R}^+_0\}$. For this is enough take the metric space $(\mathbb{R}^+, d)$ with usual metric $d(x, y) = |x - y|$, $x, y \in \mathbb{R}^+$ and $\prec = \leq$.

Proposition 2.1. *Suppose that $\{T, X, A, S, T_0\}$ is a hybrid dynamical system. Let $M \subset A$ be an invariant subset for S, and let x be any fixed point in M. Let $\{\mathbb{R}^+, X, A, \tilde{S}, \mathbb{R}_0^+\}$ be the embedding of $\{T, X, A, S, T_0\}$ from T to $\mathbb{R}^+$ with respect to x. Then M is also an invariant subset for system $\tilde{S}$, (S, M) and $(\tilde{S}, M)$ possess identical stability properties and S and $\tilde{S}$ have identical boundedness properties.*

Proof. See [5]. □

2.4. An example

Consider the following problem of initial value

$$\begin{cases} \dot{x}(t) - bx(t) = 0 \\ \\ x(0) = x_0 \geq 0 \end{cases},$$

where $x \in C[\mathbb{R}^+, \mathbb{R}]$ and $b < 0$. This differential equation determine the dynamical system $\{T, X, A, S, T_0\} = \{\mathbb{R}^+, \mathbb{R}^+, \{x_0\}, S, 0\}$, with $S = \{p(., x_0, 0) : \mathbb{R}^+ \to \mathbb{R}^+\}$ such that $p(t, x_0, 0) = x_0 e^{bt}$. We have the follows results:
(i) $(S, \{0\})$ is invariant, since $p(t, x_0, 0) = p(t, 0, 0) = 0e^{bt} = 0 \in \{0\}$ for all $t \in \mathbb{R}^+$.
(ii) For every $\epsilon > 0$, let $0 < \delta = \epsilon$ and then $d(p(t, x_0, 0), \{0\}) = x_0 e^{bt} < \delta e^{bt} = \epsilon e^{bt} \leq \epsilon$, whenever $d(x_0, \{0\}) = x_0 < \delta$. Therefore $(S, \{0\})$ is uniformly stable.
(iii) $(S, \{0\})$ is asymptotically estable, since if $d(x_0, \{0\}) = x_0 < \eta < 1$ then $\lim_{t\to\infty} d(p(t, x_0, 0), \{0\}) = \lim_{t\to\infty} p(t, x_0, 0) = \lim_{t\to\infty} x_0 e^{bt} = 0$.
(iv) Given $\epsilon > 0$, for $\alpha > 0$ and $\alpha > -b$, let $\delta = \delta(\epsilon) > 0$ such that $\delta = \epsilon$. Then, $d(p(t, x_0, 0), \{0\}) = p(t, x_0, 0) = x_0 e^{bt} < \epsilon e^{bt} \leq \epsilon e^{-\alpha t}$, whenever $d(x_0, \{0\}) = x_0 < \delta$. Thus $(S, \{0\})$ is exponentially estable, and then uniformly asymptotically estable.

3. Principal Lyapunov Theorems

To follow we presented two Principal Lyapunov Theorems for the class of discontinuous dynamical systems.

Definition 3.1. *We will call a dynamical system defined on $\mathbb{R}^+$ whose motions are not continuous with respect to time a discontinuous dynamical system.*

Theorem 3.1. *Let $\{\mathbb{R}^+, (X, d), A, S, \mathbb{R}_0^+\}$ be a discontinuous dynamical system and let $M \subset A$ be closed. Assume that there exists a function $V : X \times \mathbb{R}^+ \to \mathbb{R}^+$ and functions $\psi_1, \psi_2 \in K\mathbb{R}$ such that*

$$\psi_1(d(x, M)) \leq V(x, t) \leq \psi_2(d(x, M)),$$

for all $x \in X$ and $t \in \mathbb{R}^+$.
(a) Assume that for every $p(., a, \tau_0^p) \in S$, $V(p(t, a, \tau_0^p), t)$ is continuous on $\mathbb{R}^+_{\tau_0^p} = \{t \in \mathbb{R}^+ : t \geq \tau_0^p\}$ except on a set of discontinuities $E_{V(p)} \subset E_p$, where $E_p = \{\tau_0^p, \tau_1^p, ... : 0 \leq \tau_0^p < \tau_1^p < ...\}$ is the set of points of discontinuities of $p(., a, \tau_0^p)$.

Also, assume that there exists a neighborhood $U \subset A$ of M such that $V(p(t, a, \tau_0^p), t)$ is nonincreasing for all $a \in U$ and all $t \geq \tau_0^p$, and assume that there exists a increasing function $h \in C[\mathbb{R}^+, \mathbb{R}^+]$, with $h(0) = 0$ such that

$$V(p(t, a, \tau_0^p), t) \leq h(V(p(\tau_k^p, a, \tau_0^p), \tau_k^p)), t \in (\tau_k^p, \tau_{k+1}^p).$$

Then (S, M) is invariant and uniformly stable.
(b) If in addition to the assumptions given in (a) there exists a function $\psi_3 \in K$ defined on $\mathbb{R}^+$ such that

$$DV(p(\tau_k^p, a, \tau_0^p), \tau_k^p) \leq -\psi_3(d(p(\tau_k^p, a, \tau_0^p), M)),$$

for all $a \in U$, $k \in \mathbb{N}$, where

$$DV(p(\tau_k^p, a, \tau_0^p), \tau_k^p) :=$$

$$\frac{1}{\tau_{k+1}^p - \tau_k^p}\left[V(p(\tau_{k+1}^p, a, \tau_0^p), \tau_{k+1}^p) - V(p(\tau_k^p, a, \tau_0^p), \tau_k^p)\right] \tag{3.1}$$

then (S, M) is uniformly asymptotically stable.

Proof. (a) We will prove that (S, M) is invariant. If $a \in M$, then $V(p(\tau_0^p, a, \tau_0^p), \tau_0^p) = 0$ since $V(p(\tau_0^p, a, \tau_0^p), \tau_0^p) = V(a, \tau_0^p) \leq \psi_2(d(a, M))$ and $d(a, M) = 0$. Therefore, we know that $V(p(\tau_k^p, a, \tau_0^p), \tau_k^p) = 0$ for all $k \geq 0$ since $V(p(\tau_k^p, a, \tau_0^p), \tau_k^p)$ is nonincreasing for all $a \in M$ and $\tau_{k+1}^p \geq \tau_0^p$ and $V(X \times \mathbb{R}^+) \subset \mathbb{R}^+$. Furthermore $V(p(t, a, \tau_0^p), t) = 0$ for all $t \geq \tau_0^p$ since $V(p(t, a, \tau_0^p), t) \leq h(V(p(\tau_k^p, a, \tau_0^p), \tau_k^p)) = h(0) = 0$. This implies that $\psi_1(d(p(t, a, \tau_0^p), M)) \leq V(p(t, a, \tau_0^p), t) = 0$ and then $d(p(t, a, \tau_0^p), M) = 0$, that is, $p(t, a, \tau_0^p) \in M$ for all $t \geq \tau_0^p$, since M is closed. Therefore (S, M) is invariant.
Since h is continuous and $h(0) = 0$, then for any $\epsilon > 0$ there exists $\delta = \delta(\epsilon) > 0$ such that $h(y) < \psi_1(\epsilon)$ as long as $0 \leq y \leq \delta$. We can assume that $\delta < \psi_1(\epsilon)$. Thus, for any motion $p(., a, \tau_0^p) \in S$, as long as the initial condition $d(a, M) < \psi_2^{-1}(\delta)$ is satisfied, with $a \in U$, it follows that $V(p(\tau_0^p, a, \tau_0^p), \tau_0^p) = V(a, \tau_0^p) \leq \psi_2(d(a, M)) < \psi_2(\psi_2^{-1}(\delta)) = \delta$ and $V(p(\tau_k^p, a, \tau_0^p), \tau_k^p) < \delta$ for all k, since $V(p(\tau_k^p, a, \tau_0^p), \tau_k^p)$ is nonincreasing and then

$$d(p(\tau_k^p, a, \tau_0^p), M) \leq \psi_1^{-1}(V(p(\tau_k^p, a, \tau_0^p), \tau_k^p)) < \psi_1^{-1}(\delta) < \epsilon \ .$$

Furthermore, for any $t \in (\tau_k^p, \tau_{k+1}^p)$, we can conclude that

$$V(p(t, a, \tau_0^p), t) \leq h(V(p(\tau_k^p, a, \tau_0^p), \tau_k^p)) \leq h(\delta) < \psi_1(\epsilon)$$

and

$$d(p(t, a, \tau_0^p), M) \leq \psi_1^{-1}(V(p(t, a, \tau_0^p), t)) < \psi_1^{-1}(\psi_1(\epsilon)) = \epsilon.$$

Therefore (S, M) is uniformly stable.
(b) Letting $z_k^p = V(p(\tau_k^p, a, \tau_0^p), \tau_k^p)$, with $a \in U$, we obtain from the assumptions of the theorem that

$$-(\tau_{k+1}^p - \tau_k^p)\psi_3(\psi_2^{-1}(z_k^p)) = -(\tau_{k+1}^p - \tau_k^p)\psi_3(\psi_2^{-1}(V(p(\tau_k^p, a, \tau_0^p), \tau_k^p))) \geq$$

$$-(\tau_{k+1}^p - \tau_k^p)\psi_3(d(p(\tau_k^p, a, \tau_0^p), M)) \geq$$
$$V(p(\tau_{k+1}^p, a, \tau_0^p), \tau_{k+1}^p) - V(p(\tau_k^p, a, \tau_0^p), \tau_k^p) = z_{k+1}^p - z_k^p$$

If we denote $\psi = \psi_3 \circ \psi_2^{-1}$, then $\psi \in K$ and the last inequality becomes

$$z_{k+1}^p - z_k^p \leq -(\tau_{k+1}^p - \tau_k^p)\psi(z_k^p).$$

It follows that

$$\psi(z_k^p) \leq \frac{z_k^p - z_{k+1}^p}{\tau_{k+1}^p - \tau_k^p} \leq \frac{z_0^p - z_{k+1}^p}{\tau_{k+1}^p - \tau_k^p} \leq \frac{z_0^p}{\tau_{k+1}^p - \tau_k^p}. \tag{3.2}$$

Now, consider a fixed $\delta > 0$. For any $\epsilon > 0$, we can choose a $\tau > 0$ such that

$$\max\{\psi_1(\psi^{-1}(\frac{\psi_2(\delta)}{\tau})), \psi_1^{-1}(h(\psi^{-1}(\frac{\psi_2(\delta)}{\tau})))\} < \epsilon$$

and $\tau_{k+1}^p - \tau_k^p > \tau$ for all k. Let $a \in U \subset A$ with $d(a, M) < \delta$ and $\tau_0^p \in \mathbb{R}_0^+$ any. For any $t \geq \tau_0^p + \tau$, t must belong to some interval $[\tau_k^p, \tau_{k+1}^p)$ for some k. It follows from (3.2) that

$$\psi(z_k^p) \leq \frac{z_0^p}{\tau_{k+1}^p - \tau_k^p} = \frac{V(p(\tau_0^p, a, \tau_0^p), \tau_0^p)}{\tau_{k+1}^p - \tau_k^p} = \frac{V(a, \tau_0^p)}{\tau_{k+1}^p - \tau_k^p} <$$
$$< \frac{V(a, \tau_0^p)}{\tau} \leq \frac{\psi_2(d(a, M))}{\tau} < \frac{\psi_2(\delta)}{\tau}$$

which implies that

$$z_k^p = V(p(\tau_k^p, a, \tau_0^p), \tau_k^p) < \psi^{-1}(\frac{\psi_2(\delta)}{\tau}) \tag{3.3}$$

and

$$V(p(t, a, \tau_0^p), t) \leq h(V(p(\tau_k^p, a, \tau_0^p), \tau_k^p)) \leq h(\psi^{-1}(\frac{\psi_2(\delta)}{\tau})) \tag{3.4}$$

if $t \in (\tau_k^p, \tau_{k+1}^p)$. In the case when $t = \tau_k^p$, it follows from (3.3) that

$$d(p(\tau_k^p, a, \tau_0^p), M) \leq \psi_1^{-1}(V(p(\tau_k^p, a, \tau_0^p), \tau_k^p)) < \psi_1^{-1}(\psi^{-1}(\frac{\psi_2(\delta)}{\tau})) < \epsilon.$$

In the case when $t \in (\tau_k^p, \tau_{k+1}^p)$, we can conclude from (3.4) that

$$d(p(t, a, \tau_0^p), M) \leq \psi_1^{-1}(V(p(t, a, \tau_0^p), t)) \leq \psi_1^{-1}(h(\psi^{-1}(\frac{\psi_2(\delta)}{\tau}))) < \epsilon.$$

This prove that (S, M) is uniformly asymptotically stable. □

Theorem 3.2. *Let $\{\mathbb{R}^+, (X, d), A, S, \mathbb{R}_0^+\}$ be a discontinuous dynamical system and let $M \subset A$ be closed. Assume that there exists a function $V : X \times \mathbb{R}^+ \to \mathbb{R}^+$ and positive constants c_1, c_2 and b such that*

$$c_1(d(x, M))^b \leq V(x, t) \leq c_2(d(x, M))^b,$$

for all x in some neighborhood X_1 of M and $t \in \mathbb{R}^+$.
(i) Assume that for every $p(., a, \tau_0^p) \in S$, $V(p(t, a, \tau_0^p), t)$ is continuous on $\mathbb{R}^+_{\tau_0^p} = \{t \in \mathbb{R}^+ : t \geq \tau_0^p\}$ except on a set of discontinuities $E_{V(p)} \subset E_p$, where $E_p = \{\tau_0^p, \tau_1^p, ... : 0 \leq \tau_0^p < \tau_1^p < ...\}$ is the set of points of discontinuities of $p(., a, \tau_0^p)$. Moreover, assume that there exists a function $h \in C[\mathbb{R}^+, \mathbb{R}^+]$, with $h(0) = 0$ such that

$$V(p(t, a, \tau_0^p), t) \leq h(V(p(\tau_k^p, a, \tau_0^p), \tau_k^p)),$$

for $t \in (\tau_k^p, \tau_{k+1}^p)$, and such that for some positive constant q, h satisfies

$$\lim_{r \to 0} \frac{h(r)}{|r|^q} = 0.$$

(ii) Suppose that there exists a constant $c_3 > 0$ such that

$$DV(p(\tau_k^p, a, \tau_0^p), \tau_k^p) \leq -c_3[d(p(\tau_k^p, a, \tau_0^p), M)]^b,$$

for all $a \in X_1$ and $k \in \mathbb{N}$, where $DV(p(\tau_k^p, a, \tau_0^p), \tau_k^p)$ is defined in (3.1).
Then (S, M) is exponentially stable.

Proof. See [5]. □

4. Converse Theorem

Now we presented a converse theorem of the theorem 3.2 in the following sense:

Theorem 4.1. *Let $\{\mathbb{R}^+, (X, d), A, S, \mathbb{R}_0^+ = \mathbb{R}^+\}$ be a discontinuous dynamical system and let $M \subset A$ be a closed invariant set, where A is a neighborhood of M. Suppose that :*
(i) every $p(., a, \tau_0^p) \in S$ is continuous everywhere on $[\tau_0^p, \infty)$, except on a set $E_p = \{\tau_0^p, \tau_1^p, ... : \tau_0^p < \tau_1^p < ...\}$, being $l = \inf_{p \in S}\{\tau_{k+1}^p - \tau_k^p\} > 0$ and $L = \sup_{p \in S}\{\tau_{k+1}^p - \tau_k^p\} < \infty$;
(ii) for any $p(., a, \tau_0^p) \in S$ is true that $p(t', p(t, a, \tau_0^p), t) = p(t', a, \tau_0^p)$ for all $t \in \mathbb{R}_0^+$ and $t' \geq t$. Furthermore, $p(t', a, t) \notin M$ if $a \notin M$, therefore $d(p(t', a, t), M) > 0$ for all $t' \geq t$ if $a \notin M$, since M is closed.
Let (S, M) be exponentially stable. Then there exists a neighborhood X_1 of M such that $X_1 \subset A$, and a mapping $V : X_1 \times \mathbb{R}^+ \to \mathbb{R}^+$ which satisfies the following conditions:
(a)there exist $\psi_1, \psi_2 \in K$ defined on $\mathbb{R}^+$ such that

$$\psi_1(d(x, M)) \leq V(x, t) \leq \psi_2(d(x, M)),$$

for all $(x, t) \in X_1 \times \mathbb{R}^+$.
(b) there exists a constant $c > 0$ such that for every $p(., a, \tau_0^p) \in S$,

$$DV(p(\tau_k^p, a, \tau_0^p), \tau_k^p) \leq -cV(p(\tau_k^p, a, \tau_0^p), \tau_k^p),$$

for $k \in \mathbb{N}$, *where* $a \in X_1$.
(c) there exists a function $h \in C[\mathbb{R}^+, \mathbb{R}^+]$, *with* $h(0) = 0$ *and* $lim_{\theta \to 0^+} \frac{h(\theta)}{\theta^q} = 0$ *for some constant* $q > 0$, *such that*

$$V(p(t, a, \tau_0^p), t) \leq h(V(p(\tau_k^p, a, \tau_0^p), \tau_k^p),$$

for every $p(., a, \tau_0^p) \in S$, $t \in (\tau_k^p, \tau_{k+1}^p)$, $a \in X_1$ *and* $\tau_0^p \in \mathbb{R}^+$.

Proof. Since (S, M) exponentially stable, for every $\epsilon > 0$ there exists $\alpha > 0$ and $r_0 = r_0(\epsilon)$ such that $d(p(t, a, \tau_0^p), M) < \epsilon e^{-\alpha(t-\tau_0^p)}$ for all $p(., a, \tau_0^p) \in S$ and all $t \in T_{a,\tau_0^p}$, whenever $d(a, M) < r_0$. Let $\phi \in K$ defined on $[0, r_0]$ such that $\phi(d(a, M)) \geq \epsilon$ if $a \notin M$, then $d(p(t, a, \tau_0^p), M) < \epsilon e^{-\alpha(t-\tau_0^p)} \leq \phi(d(a, M))e^{-\alpha(t-\tau_0^p)}$. If $a \in M$, $p(t, a, \tau_0^p) \in M$ therefore $d(p(t, a, \tau_0^p), M) = 0 \leq \phi(d(a, M))e^{-\alpha(t-\tau_0^p)} = 0$, thus

$$d(p(t, a, \tau_0^p), M) \leq \phi(d(a, M))e^{-\alpha(t-\tau_0^p)}. \tag{4.1}$$

Let $X_1 = \{x \in A : d(x, M) < r_0\}$. For $(x, t) \in X_1 \times \mathbb{R}^+$, define

$$V(x, t) = \sup_{t' \geq t}\{d(p(t', x, t), M).e^{\alpha(t'-t)}\}.$$

For $a \in X_1$ and $\tau_0^p \in \mathbb{R}^+$, we have

$$\begin{aligned} V(p(t, a, \tau_0^p), t) &= \sup_{t' \geq t}\{d(p(t', p(t, a, \tau_0^p), t), M).e^{\alpha(t'-t)}\} \\ &= \sup_{t' \geq t}\{d(p(t', a, \tau_0^p), M).e^{\alpha(t'-t)}\} \end{aligned} \tag{4.2}$$

therefore

$$\begin{aligned} V(p(\tau_{k+1}^p, a, \tau_0^p), \tau_{k+1}^p) &= \sup_{t' \geq \tau_{k+1}^p}\{d(p(t', a, \tau_0^p), M)e^{\alpha(t'-\tau_{k+1}^p)}\} \\ &= \sup_{t' \geq \tau_{k+1}^p}\{d(p(t', a, \tau_0^p), M)e^{\alpha(t'-\tau_k^p)}e^{-\alpha(\tau_{k+1}^p-\tau_k^p)}\} \\ &= e^{-\alpha(\tau_{k+1}^p-\tau_k^p)} \sup_{t' \geq \tau_{k+1}^p}\{d(p(t', a, \tau_0^p), M)e^{\alpha(t'-\tau_k^p)}\} \\ &\leq e^{-\alpha l} \sup_{t' \geq \tau_{k+1}^p}\{d(p(t', a, \tau_0^p), M)e^{\alpha(t'-\tau_k^p)}\} \\ &\leq e^{-\alpha l} \sup_{t' \geq \tau_k^p}\{d(p(t', a, \tau_0^p), M)e^{\alpha(t'-\tau_k^p)}\} = e^{-\alpha l}V(p(\tau_k^p, a, \tau_0^p), \tau_k^p). \end{aligned}$$

Letting $c = (1/L)(1 - e^{-\alpha l})$, we obtain

$$\begin{aligned} DV(p(\tau_k^p, a, \tau_0^p), \tau_k^p) &= \frac{1}{\tau_{k+1}^p - \tau_k^p}\big(V(p(\tau_{k+1}^p, a, \tau_0^p), \tau_{k+1}^p) - V(p(\tau_k^p, a, \tau_0^p), \tau_k^p)\big) \\ &\leq -\frac{1}{\tau_{k+1}^p - \tau_k^p}\big(1 - e^{-\alpha l}\big)V(p(\tau_k^p, a, \tau_0^p), \tau_k^p) \end{aligned}$$

$$\leq -\frac{1}{L}\big(1-e^{-\alpha l}\big)V(p(\tau_k^p,a,\tau_0^p),\tau_k^p) = -cV(p(\tau_k^p,a,\tau_0^p),\tau_k^p).$$

Also, for all $(x,t) \in X_1 \times \mathbb{R}^+$ (4.1) imply that

$$V(x,t) = \sup_{t' \geq t}\{d(p(t',x,t),M)e^{\alpha(t'-t)}\}$$

$$\leq \sup_{t'\geq t}\{\phi(d(x,M))e^{-\alpha(t'-t)}e^{\alpha(t'-t)})\} = \sup_{t'\geq t}\{\phi(d(x,M))\} = \phi(d(x,M)).$$

Letting $\psi_2 \in K$ defined on $\mathbb{R}^+$ such that $\psi_2(r) = \phi(r)$ if $r \in [0, r_0]$, then

$$V(x,t) \leq \psi_2(d(x,M)).$$

Furthermore

$$V(x,t) = \sup_{t'\geq t}\{d(p(t',x,t),M)e^{\alpha(t'-t)}\} \geq d(p(t',x,t),M) \geq \psi_1(d(x,M))$$

for some $\psi_1 \in K$ defined on $\mathbb{R}^+$. By (4.2) we have for every $t \in (\tau_k^p, \tau_{k+1}^p)$, $\tau_0^p \in \mathbb{R}^+$ and $a \in X_1$

$$V(p(t,a,\tau_0^p),t) = \sup_{t'\geq t}\{d(p(t',a,\tau_0^p),M)e^{\alpha(t'-t)}\}$$

$$= \sup_{t'\geq t}\{d(p(t',a,\tau_0^p),M)e^{\alpha(t'-\tau_k^p)}e^{-\alpha(t-\tau_k^p)}\} \leq \sup_{t'\geq t}\{d(p(t',a,\tau_0^p),M)e^{\alpha(t'-\tau_k^p)}\}$$

$$\leq \sup_{t'\geq \tau_k^p}\{d(p(t',a,\tau_0^p),M)e^{\alpha(t'-\tau_k^p)}\} = V(p(\tau_k^p,a,\tau_0^p),\tau_k^p).$$

Letting $h \in C[\mathbb{R}^+,\mathbb{R}^+]$ such that $h(r) = r$ and $q = \frac{1}{2}$, it follows that $\lim_{\theta\to 0^+}\frac{h(\theta)}{\theta^q} = \frac{\theta}{\theta^{\frac{1}{2}}} = 0$ and

$$V(p(t,a,\tau_0^p),t) \leq V(p(\tau_k^p,a,\tau_0^p),\tau_k^p) = h(V(p(\tau_k^p,a,\tau_0^p),\tau_k^p)).$$

$\square$

Resumo. Neste trabalho introduzimos um modelo geral para os Sistemas Dinâmicos Híbridos e para tais sistemas introduzimos o conceito usual de estabilidade de Lyapunov. Além disso, estabelecemos dois Teoremas Principais de Lyapunov e um teorema de conversão.

References

[1] L. Hou, “Qualitative Analysis of Discontinuous Deterministic and Stochastic Dynamical Systems”, Ph.D. Thesis, University of Notre Dame, Indiana, 2000.

[2] A.N. Michel, Recent trends in the stability analysis of hybrid dynamical systems, *IEEE Transactions on Circuits and Systems*, **45**, No.1 (1999), 120-134.

[3] A.N. Michel, L. Hou, Stability analysis of a general class of hybrid dynamical systems, in "Proceedings of the American Control Conference", pp. 2805-2809, Albuquerque, New Mexico, 1997.

[4] A.N. Michel, B. Hu, Towards a stability theory of general hybrid dynamical systems, *Automatica*, **35** (1999), 371-384.

[5] A.N. Michel, B. Hu, K. Wang, "Qualitative Theory of Dynamical Systems", Marcel Dekker, New York, 2001.

[6] H. Ye, A.N. Michel, Stability theory for hybrid dynamical systems, *IEEE Transactions on Automatic Control*, **43**, No.4 (1998), 461-474.

[7] H. Ye, A.N. Michel, A general model for the qualitative analysis of hybrid dynamical systems, in "Proceedings of the 34th Conference on Decision and Control", pp. 1473-1477, IEEE, New Orleans, 1995.