

On the Construction of Spherical Designs

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Abstract. We study special subsets of the unit sphere in \mathbb{R}^m , $m \geq 2$, the so-called *spherical designs* in the literature. Among other things we introduce a new equivalence for the concept and investigate the construction of designs through rotations of \mathbb{R}^m and projections over the equator of the sphere.

1. Introduction

The standard definition of spherical design requires that certain equally weighted cubature formulas be exact for spherical polynomials up to a certain degree. Let us be a little bit more specific. If S^{m-1} denotes the unit sphere in \mathbb{R}^m and q is a positive integer, a finite nonempty subset W of S^{m-1} is said to be a *q-design* if

$$\frac{1}{\sigma_{m-1}} \int_{S^{m-1}} f(x) d\sigma(x) = \frac{1}{|W|} \sum_{x \in W} f(x), \quad f \in \mathcal{P}_q. \quad (1.1)$$

Here, σ_{m-1} stands for the surface area of S^{m-1} , the integration is the usual one over S^{m-1} , $|W|$ is the cardinality of W and \mathcal{P}_q is the set of all polynomials in m variables having degree at most q . The concept has applications in code theory, discrepancy and combinatorics, approximation theory, Monte-Carlo methods, etc. We refer the reader to many old references authored by Goethals, Delsarte and Seidel where the concept was investigated quite well.

Since the abstract of this paper describes what the aim of the paper is, we complete the section presenting the basic material needed in the forthcoming sections. The proofs will be omitted since most of them are well known. The basic references are [2, 7, 9, 10].

We shall write \mathcal{H}_q to denote the subset of \mathcal{P}_q formed by its homogeneous elements and \mathcal{A}_q to denote the subset of \mathcal{H}_q formed by harmonic polynomials. Throughout the paper, we shall be dealing with the elements of these sets restricted to S^{m-1} . The notation for the resulting restricted spaces will be the same. Usually, the elements of \mathcal{H}_q restricted to S^{m-1} are called m -dimensional spherical harmonics of degree q .

The basic metric structure used in the paper is that of $L^2(S^{m-1})$ as explained in the references quoted above. As so, the sets \mathcal{A}_q become orthogonal subspaces of

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$L^2(S^{m-1})$ for different values of q . Every element p of \mathcal{H}_q has a decomposition in the form

$$p = \sum_{j=0}^{\lfloor q/2 \rfloor} h_{q-2j}, \quad h_{q-2j} \in \mathcal{A}_{q-2j}, \quad j = 0, 1, \dots, \lfloor q/2 \rfloor. \quad (1.2)$$

In some passages of the text we shall deal with the Legendre polynomial R_q^m of degree q associated with the integer m , as described in [7]. It is normalized by the relation $R_q^m(1) = 1$. If $m \geq 3$, it relates to the Gegenbauer polynomial $P_q^{(m-2)/2}$ of degree q associated with the dimension $(m-2)/2$ as follows ([7, 11]):

$$R_q^m = \frac{(m-3)!q!}{(q+m-3)!} P_q^{(m-2)/2}. \quad (1.3)$$

The orthogonality of these polynomials refers to the inner product

$$\langle f, g \rangle' := \int_{-1}^1 f(t)g(t) (1-t^2)^{(m-3)/2} dt. \quad (1.4)$$

The following formula, known as the addition formula, makes the bridge between spherical harmonics and Legendre polynomials:

$$\sum_{j=1}^{N_q} Y_j^q(x) Y_j^q(y) = \frac{N_q}{\sigma_{m-1}} R_q^m(x \cdot y), \quad x, y \in S^{m-1}. \quad (1.5)$$

Here, $\{Y_j^q : j = 1, 2, \dots, N_q^m\}$ is a fixed orthonormal basis of \mathcal{A}_q and \cdot is the usual inner product of \mathbb{R}^m . An induction process shows that it is possible to find a fundamental system for \mathcal{A}_q , that is, a subset $\{x_1, x_2, \dots, x_{N_q^m}\}$ of S^{m-1} for which the matrix $(f_j(x_i))$ is invertible for every basis $\{f_1, f_2, \dots, f_{N_q^m}\}$ of \mathcal{A}_q .

The Funk-Hecke formula states that

$$\int_{S^{m-1}} K(x \cdot y) h(y) d\sigma(y) = \alpha_q(K) h(x), \quad x \in S^{m-1}, \quad (1.6)$$

whenever $h \in \mathcal{A}_q$, K is a function with domain $[-1, 1]$ and the integral makes sense. The coefficient $\alpha_q(K)$ is calculated through the formula

$$\alpha_q(K) := \sigma_{m-2} \int_{-1}^1 K(t) R_q^m(t) (1-t^2)^{(m-3)/2} dt. \quad (1.7)$$

An immediate consequence of the formula is the reproducing property

$$h(x) = \frac{N_q}{\sigma_{m-1}} \int_{S^{m-1}} R_q^m(x \cdot y) h(y) d\sigma(y), \quad x \in S^{m-1}, \quad h \in \mathcal{A}_q. \quad (1.8)$$

Not so popular is the addition formula for Gegenbauer polynomials ([1]). For real numbers θ, ϕ and ϑ , it reads

$$\begin{aligned} P_q^{(m-2)/2}(\cos \theta \cos \phi + \sin \theta \operatorname{sen} \phi \cos \vartheta) &= P_q^{(m-2)/2}(\cos \theta) P_q^{(m-2)/2}(\cos \phi) \\ &+ \sum_{i=1}^q \beta_{i,q} Q_i(\theta) Q_i(\phi) P_i^{(m-3)/2}(\cos \vartheta), \end{aligned}$$

where $\beta_{i,q} > 0, i = 1, 2, \dots, q$ and

$$Q_i(\theta) = \text{sen}^i \theta P_{q-i}^{(m-2+2i)/2}(\cos \theta), \quad i = 1, 2, \dots, q, \quad \theta \in \mathbb{R}. \tag{1.9}$$

Finally, we recall Dougall's linearization formula [1, 3, 8]: If k and n are non-negative integers, then

$$R_k^m R_n^m = \sum_{j=0}^{k \wedge n} \alpha_j^{k,n} R_{k+n-2j}^m, \tag{1.10}$$

where every coefficient $\alpha_j^{k,n}$ is positive.

2. Equivalences for the Concept of Design

The concept of design has many equivalent formulations. The definitions of design on its own implies the following facts: a nonempty subset W of S^{m-1} is a 1-design if and only if $\sum_{x \in W} x = 0$. If $W \subset S^{m-1}$ is a q -design and $q_1 \in \{1, 2, \dots, q\}$ then W is a q_1 -design. Below, we recall the most recent equivalence for the concept, due to Yudin ([12]). It allows us to present nontrivial examples of designs in an elementary way.

Theorem 2.1. *Let W be a finite nonempty subset of S^{m-1} . The following assertions are equivalent:*

- (i) W is a q -design;
- (ii) $\sum_{x \in W} R_k^m(x \cdot y) = 0, y \in S^{m-1}, k = 1, 2, \dots, q;$
- (iii) It holds

$$\frac{1}{|W|} \sum_{x \in W} (x \cdot y)^k = \alpha_k, \quad y \in S^{m-1}, \quad k = 1, 2, \dots, q, \tag{2.1}$$

where

$$\alpha_k = \begin{cases} 0, & k \text{ odd} \\ \frac{\sigma_{m-2}}{\sigma_{m-1}} \int_{-1}^1 t^k (1-t^2)^{(m-3)/2} dt, & k \text{ even.} \end{cases} \tag{2.2}$$

The Funk-Hecke formula shows that the integral appearing in the previous formula can be re-written as

$$\frac{1}{\sigma_{m-1}} \int_{S^{m-1}} (y \cdot x)^k d\sigma(x), \tag{2.3}$$

where y is an arbitrary point of S^{m-1} .

The set $W_1 = \{(1, 0), (0, 1), (-1, 0), (0, -1)\}$ is a 3-design in S^1 . Indeed, we verify that W_1 satisfies Condition (iii) above. Let $y = (y_1, y_2) \in S^1$. If $k = 1$, the sum in (iii) takes the form $y_1 - y_1 + y_2 - y_2 = 0$. If $k = 2$, it reduces itself to $2(y_1^2 + y_2^2) := 2\|y\|^2$. On the other hand, taking $y = (1, 0)$ in (2.3) and using polar coordinates we obtain

$$\alpha_2 = \frac{1}{2\pi} \int_{S^1} ((1, 0) \cdot (y_1, y_2))^2 d\sigma(y) = \frac{1}{2\pi} \int_0^{2\pi} \cos^2 t dt = \frac{1}{2}.$$

Finally, if $k = 3$, $y_1^3 + (-y_1)^3 + y_2^3 + (-y_2)^3 = 0$. From Theorem 2.1, the desired conclusion follows.

The set $W_2 = \{(1, 0, 0), (0, 1, 0), (0, 0, 1), (-1, 0, 0), (0, -1, 0), (0, 0, -1)\}$ is a 3-design in S^2 . To see that we verify that W_2 satisfies Condition (ii) in Theorem 2.1. The low degree Legendre polynomials are given by $R_1^3(t) = t$, $2R_2^3(t) = 3t^2 - 1$ and $2R_3^3(t) = 5t^3 - 3t$. Fixing $y = (y_1, y_2, y_3) \in S^2$ we have the following possibilities: if $k = 1$,

$$\sum_{x \in W} R_1^3(x \cdot y) = y_1 + y_2 + y_3 - y_1 - y_2 - y_3 = 0;$$

if $k = 2$,

$$\sum_{x \in W} R_2^3(x \cdot y) = \frac{1}{6}(6y_1^2 + 6y_2^2 + 6y_3^2 - 6) = 3\|y\|^2 - 3 = 0;$$

Finally, if $k = 3$,

$$\sum_{x \in W} R_3^3(x \cdot y) = \frac{1}{2}(5y_1^3 + 5y_2^3 + 5y_3^3 - 5y_1^3 - 5y_2^3 - 5y_3^3 - 3y_1 - 3y_2 - 3y_3 + 3y_1 + 3y_2 + 3y_3) = 0.$$

Another criteria for a set W to be a q -design are given below. We include a proof here just because we were unable to find one in the literature. Arguments similar to the ones used in the proof can be found in [4,5,12]. We write \mathcal{O}_m to denote the set of all orthogonal transformations on \mathbb{R}^m .

Theorem 2.2. *Let W be a finite nonempty subset of S^{m-1} . The following assertions are equivalent:*

- (i) W is a q -design;
- (ii) It holds

$$\frac{1}{|W|} \sum_{x \in W} p(\rho(x)) = \frac{1}{\sigma_{m-1}} \int_{S^{m-1}} p(x) d\sigma(x), \quad p \in \mathcal{H}_k, \quad k = 0, 1, \dots, q, \quad \rho \in \mathcal{O}_m.$$

- (iii) $\sum_{x \in W} h(x) = 0$, $h \in \mathcal{A}_k$, $k = 1, 2, \dots, q$;
- (iv) $\sum_{x \in W} p(\rho(x)) = \sum_{x \in W} p(x)$, $p \in \mathcal{H}_k$, $k = 0, 1, \dots, q$, $\rho \in \mathcal{O}_m$.

Proof. Let $k \in \{0, 1, \dots, q\}$, $p \in \mathcal{H}_k$ and $\rho \in \mathcal{O}_m$. If (i) holds, the invariance of the Lebesgue measure with respect of elements of \mathcal{O}_m on S^{m-1} and the fact that $p \circ \rho \in \mathcal{P}_q$ justify the equalities

$$\sum_{x \in W} p(\rho(x)) = \frac{1}{\sigma_{m-1}} \int_{S^{m-1}} p(x) d\sigma(\rho^*(x)) = \frac{1}{\sigma_{m-1}} \int_{S^{m-1}} p(x) d\sigma(x), \quad (2.4)$$

where ρ^* is the adjoint of ρ . Thus, (ii) follows. To verify that the later implies (iii), it suffices to take ρ as the identity transformation and to observe that $\mathcal{A}_k \subset \mathcal{H}_k$. Next, assume (iii) holds. Let $k \in \{1, 2, \dots, q\}$, $p \in \mathcal{H}_k$ e $\rho \in \mathcal{O}_m$. Decomposing p according to (1.2), we shall establish (iv) in two steps. If k is odd, then $k - 2j \geq 1$, $j = 0, 1, \dots, \lfloor k/2 \rfloor$ and, consequently,

$$\sum_{x \in W} p(x) = \sum_{j=0}^{\lfloor k/2 \rfloor} \sum_{x \in W} h_{k-2j}(x) = 0. \quad (2.5)$$

Since $h_{k-2j} \circ \rho \in \mathcal{A}_k$, $j = 0, 1, \dots, \lfloor k/2 \rfloor$, a similar procedure leads to the formula $\sum_{x \in W} p(\rho(x)) = 0$. If k is even, we proceed in the same way to obtain

$$\sum_{x \in W} p(x) = |W| = \sum_{x \in W} p(\rho(x)). \tag{2.6}$$

In both cases, we are led to the equality in (iv). To finish the proof, we show that (iv) implies (iii) in Theorem 2.1. Let $k \in \{0, 1, \dots, q\}$. For every $y \in S^{m-1}$, let $\rho_y \in \mathcal{O}_m$ be chosen so that the coordinates of y with respect to the canonical basis of \mathbb{R}^m be, respectively, the entries in the first row in the matrix representation of ρ_y with respect to that same basis. Then,

$$\sum_{x \in W} (x \cdot y)^k = \sum_{x \in W} p(\rho_y(x)), \tag{2.7}$$

where $p(x) = x_1^k$ ($x_1 =$ the first component of x). If (iv) holds, we conclude that

$$\sum_{x \in W} (x \cdot y)^k = \sum_{x \in W} p(x). \tag{2.8}$$

Thus, the function $y \in S^{m-1} \rightarrow \sum_{x \in W} (x \cdot y)^k$ is a constant β , not depending on y . However, it is easily seen that

$$\beta = \frac{1}{\sigma_{m-1}} \int_{S^{m-1}} \beta d\sigma(y) = \sum_{x \in W} \frac{1}{\sigma_{m-1}} \int_{S^{m-1}} (x \cdot y)^k d\sigma(y). \tag{2.9}$$

It is now clear that $\beta = |W|\alpha_k$, where α_k is given in (2.2). □

3. The Results

The following nice equivalence criteria for the concept of design is a consequence of the results in the previous section. As far as we know, it has not been published yet.

Theorem 3.1. *Let W be a finite nonempty subset of S^{m-1} . The following assertions are equivalent:*

- (i) W is a q -design;
- (ii) $\sum_{x \in W} R_k^m(\rho(x) \cdot y) = \sum_{x \in W} R_k^m(x \cdot y)$, $k = 1, 2, \dots, q$, $y \in S^{m-1}$, $\rho \in \mathcal{O}_m$.

Proof. Since every function $x \in S^{m-1} \rightarrow R_k^m(x \cdot y)$, $y \in S^{m-1}$ is an element of \mathcal{H}_k , one implication follows directly from the previous theorem. As for the other, it suffices to verify that (ii) implies Condition (iv) in Theorem 2.2. To do that, in view of (1.2), it suffices to verify that condition in the case where $h \in \mathcal{A}_k$, $k = 0, 1, \dots, q$. The case $k = 0$ is trivial. As for the others, fix $k \in \{1, 2, \dots, q\}$ and let $h \in \mathcal{A}_k$. Observing that $\alpha_k(R_k^m) > 0$, the Funk-Hecke formula leads to

$$\begin{aligned} \sum_{x \in W} h(x) &= \frac{1}{\alpha_k(R_k^m)} \sum_{x \in W} \int_{S^{m-1}} R_k^m(x \cdot y) h(y) d\sigma(y) \\ &= \frac{1}{\alpha_k(R_k^m)} \int_{S^{m-1}} \left(\sum_{x \in W} R_k^m(x \cdot y) \right) h(y) d\sigma(y). \end{aligned}$$

If (ii) holds, the equality

$$\sum_{x \in W} h(x) = \sum_{x \in W} h(\rho(x)), \quad \rho \in \mathcal{O}_m, \tag{3.1}$$

follows from the previous formula. □

To proceed, we assume that $m \geq 3$ and introduce additional notation. Making use of a fixed pole ε on S^{m-1} , we shall decompose a point $x \in S^{m-1}$ in the following form

$$x = \lambda\varepsilon + \sqrt{1 - \lambda^2} x'_\varepsilon, \quad \lambda \in [-1, 1], \quad x'_\varepsilon \in S_\varepsilon^{m-1}, \tag{3.2}$$

where S_ε^{m-1} is the equator of S^{m-1} , orthogonal to ε . The set S_ε^{m-1} is in fact a copy of S^{m-2} isometrically embedded in S^{m-1} . The embedding itself will be denoted by Ψ_ε .

Given a finite nonempty subset W of S^{m-1} , W'_ε will denote the spherical projection of W from ε onto the equator of S^{m-1} , that is, $W'_\varepsilon := \{x'_\varepsilon : x \in W\}$. In particular, $S_\varepsilon^{m-1} = \{x'_\varepsilon : x \in S^{m-1}\}$.

Theorem 3.2 below provides a concise method to construct spherical designs in S^{m-2} from spherical designs in S^{m-1} .

Theorem 3.2. *Let W be a finite nonempty subset of S^{m-1} and $\varepsilon \in S^{m-1}$. Assume there exists $\lambda \in (-1, 1) \setminus \cup_{k=0}^{q-1} \{t : R_{q-k}^{m+k}(t) = 0\}$ so that $W \subset \{x \in S^{m-1} : x \cdot \varepsilon = \lambda\}$. If W is a q -design in S^{m-1} then $\Psi_\varepsilon^{-1}(W'_\varepsilon)$ is a q -design in S^{m-2} .*

Proof. We intend to use Theorem 2.1. Using (3.2), it is easily seen that

$$\sum_{x \in W} \sum_{y \in W} R_k^m(x \cdot y) = \sum_{x \in W} \sum_{y \in W} R_k^m(\lambda^2 + (1 - \lambda^2)x'_\varepsilon \cdot y'_\varepsilon), \quad x, y \in S^{m-1}. \tag{3.3}$$

Since the map $x \in W \rightarrow x'_\varepsilon \in W'_\varepsilon$ is one-to-one, it follows that

$$\sum_{x \in W} \sum_{y \in W} R_k^m(\lambda^2 + (1 - \lambda^2)x'_\varepsilon \cdot y'_\varepsilon) = \sum_{x'_\varepsilon \in W'_\varepsilon} \sum_{y'_\varepsilon \in W'_\varepsilon} R_k^m(\lambda^2 + (1 - \lambda^2)x'_\varepsilon \cdot y'_\varepsilon). \tag{3.4}$$

If S_k denotes the very last sum in (3.4), we can use (1.3) and the addition formula for Gegenbauer polynomials to obtain

$$\begin{aligned} S_k &= \sum_{x'_\varepsilon \in W'_\varepsilon} \sum_{y'_\varepsilon \in W'_\varepsilon} \sum_{l=0}^k \beta_{l,k}^m (1 - \lambda^2)^l (R_{k-l}^{m+l}(\lambda))^2 R_l^{m-1}(x'_\varepsilon \cdot y'_\varepsilon) \\ &= \sum_{l=0}^k \beta_{l,k}^m (1 - \lambda^2)^l (R_{k-l}^{m+l}(\lambda))^2 \sum_{x'_\varepsilon \in W'_\varepsilon} \sum_{y'_\varepsilon \in W'_\varepsilon} R_l^{m-1}(x'_\varepsilon \cdot y'_\varepsilon), \end{aligned}$$

where

$$\beta_{l,k}^m = \frac{(m-3)!k!}{(k+m-3)!} \beta_{l,k}, \quad l = 0, 1, \dots, k. \tag{3.5}$$

If W is a q -design, (3.3) and Theorem 2.1 imply that $S_k = 0$, $k = 1, 2, \dots, q$. The quadratic forms appearing in the expression that calculates S_k are all nonnegative definite due to the addition formula (1.5). Being the coefficients $\beta_{l,k}^m$ and $(1 - \lambda^2)^l$ all positive, our assumption on λ shows that the previous equality is equivalent to

$$\sum_{x'_\varepsilon \in W'_\varepsilon} \sum_{y'_\varepsilon \in W'_\varepsilon} R_l^{m-1}(x'_\varepsilon \cdot y'_\varepsilon) = 0, \quad l = 0, 1, \dots, q. \tag{3.6}$$

Let x'' and y'' be points of S^{m-2} so that $\Psi_\varepsilon(x'') = x'_\varepsilon$ and $\Psi_\varepsilon(y'') = y'_\varepsilon$. Hence,

$$\begin{aligned} 0 &= \sum_{x'_\varepsilon \in W'_\varepsilon} \sum_{y'_\varepsilon \in W'_\varepsilon} R_l^{m-1}(x'_\varepsilon \cdot y'_\varepsilon) \\ &= \sum_{x'_\varepsilon \in W'_\varepsilon} \sum_{y'_\varepsilon \in W'_\varepsilon} R_l^{m-1}(x'' \cdot y'') \\ &= \sum_{x'' \in \Psi_\varepsilon^{-1}(W'_\varepsilon)} \sum_{y'' \in \Psi_\varepsilon^{-1}(W'_\varepsilon)} R_l^{m-1}(x'' \cdot y''), \quad l = 0, 1, \dots, q. \end{aligned}$$

Theorem 2.1 once again shows that $\Psi_\varepsilon^{-1}(W'_\varepsilon)$ is a q -design in S^{m-2} . □

In the last theorem of the paper, we shall deal with a design W of S^{m-1} , looking at its rim decomposition with respect to a fixed pole ε of S^{m-1} . Precisely, we shall write W as a disjoint union of the form $W = \dot{\cup}_{i=1}^n W_{\lambda_i}$, where

$$W_{\lambda_i} = W \cap \{x \cdot \varepsilon = \lambda_i : x \in S^{m-1}\}, \quad \lambda_i \in (-1, 1), \quad i = 1, 2, \dots, n, \tag{3.7}$$

and will investigate the effect of rotations on the W_{λ_i} . The result can be seen as an extension of a result proved by Yudin ([12]).

Theorem 3.3. *Let W be a finite and nonempty subset of S^{m-1} . Consider the rim decomposition $W = \dot{\cup}_{i=1}^n W_{\lambda_i}$ of W with respect to ε . Let $\rho_1, \rho_2, \dots, \rho_n$ be elements of \mathcal{O}_m so that $\rho_j(\varepsilon) = \varepsilon, j = 1, 2, \dots, n$. If W is a q -design and every $W'_{\lambda_i, \varepsilon}$ is a q_i -design, $i = 1, 2, \dots, n$, then $W_\rho := \cup_{i=1}^n \rho_i(W_{\lambda_i})$ is a ν -design, where $\nu = q \wedge q_1 \wedge \dots \wedge q_n$.*

Proof. Assuming that W is a q -design and each $W'_{\lambda_i, \varepsilon}$ is a q_i -design, we shall verify that

$$\sum_{x \in W_\rho} R_n^m(x \cdot y) = 0, \quad y \in S^{m-1}, \quad n = 1, 2, \dots, \nu. \tag{3.8}$$

If every ρ_j is the identity operator, then $W_\rho = W$. Since W is a q -design, (3.8) follows from Theorem 2.1 and the inequality $\nu \leq q$. In the general case, we write

$$\sum_{x \in W_\rho} R_n^m(x \cdot y) = \sum_{i=1}^n \sum_{x \in \rho_i(W_{\lambda_i})} R_n^m(x \cdot y), \tag{3.9}$$

and verify that the right-hand side does not depend upon $\rho_i, i = 1, 2, \dots, n$, when $n \leq \nu$. To do that, we shall look at each summand separately. Fix $i \in \{1, 2, \dots, n\}$.

Since each ρ_i preserves inner product,

$$\sum_{x \in \rho_i(W_{\lambda_i})} R_n^m(x \cdot y) = \sum_{\rho_i^*(x) \in W_{\lambda_i}} R_n^m(\rho_i^*(x) \cdot \rho_i^*(y)) = \sum_{z \in W_{\lambda_i}} R_n^m(z \cdot \rho_i^*(y)). \quad (3.10)$$

Employing the decomposition (3.2) for z and y , we deduce that

$$\begin{aligned} \sum_{x \in \rho_i(W_{\lambda_i})} R_n^m(x \cdot y) &= \sum_{z \in W_{\lambda_i}} R_n^m(\lambda_i \xi + (1 - \lambda_i^2)^{1/2} (1 - \xi^2)^{1/2} (\rho_i(z'_\varepsilon) \cdot y'_\varepsilon)) \\ &= \sum_{z'_\varepsilon \in W'_{\lambda_i}} R_n^m(\lambda_i \xi + (1 - \lambda_i^2)^{1/2} (1 - \xi^2)^{1/2} (\rho_i(z'_\varepsilon) \cdot y'_\varepsilon)), \end{aligned}$$

where $\xi = z \cdot \varepsilon$. The index arrangement in the last sum is justified by the fact that $z \in W_{\lambda_i} \rightarrow z'_\varepsilon \in W'_{\lambda_i}$ is one-to-one. Since R_n^m is a polynomial of degree n , there are coefficients $a_l(\lambda_i, \xi)$, $l = 0, 1, \dots, n$, so that

$$\sum_{x \in \rho_i(W_{\lambda_i})} R_n^m(x \cdot y) = \sum_{l=0}^n a_l(\lambda_i, \xi) (1 - \lambda_i^2)^{l/2} \sum_{z'_\varepsilon \in W'_{\lambda_i}} (z'_\varepsilon \cdot \rho_i^*(y'_\varepsilon))^l. \quad (3.11)$$

Since W'_{λ_i} is a q_i -design and $n \leq \nu$, Theorem 2.1-(iii) can be used to deduce that

$$\begin{aligned} \sum_{x \in \rho_i(W_{\lambda_i})} R_n^m(x \cdot y) &= |W'_{\lambda_i}| \sum_{\substack{l=0 \\ l \text{ par}}}^n a_l(\lambda_i, \xi) (1 - \lambda_i^2)^{l/2} \left(\frac{1}{|W'_{\lambda_i}|} \sum_{z'_\varepsilon \in W'_{\lambda_i, \varepsilon}} (z'_\varepsilon \cdot \rho_i^*(y'_\varepsilon))^l \right) \\ &= |W'_{\lambda_i}| \sum_{\substack{l=0 \\ l \text{ par}}}^n a_l(\lambda_i, \xi) (1 - \lambda_i^2)^{l/2} \alpha_l \|\rho_i^*(y'_\varepsilon)\|^l, \end{aligned}$$

It follows that

$$\sum_{x \in \rho_i(W_{\lambda_i})} R_n^m(x \cdot y) = |W'_{\lambda_i}| \sum_{\substack{l=0 \\ l \text{ par}}}^n a_l(\lambda_i, \xi) (1 - \lambda_i^2)^{l/2} \alpha_l \|y'_\varepsilon\|^l, \quad (3.14)$$

which is an expression not depending on ρ_i . Therefore, the theorem follows from the considerations at the beginning of the proof and Theorem 2.1-(ii). \square

At least two consequences of the Theorem 3.3. are easily obtained.

Corollary 3.4. *Let W be a finite nonempty subset of S^{m-1} , $\varepsilon \in S^{m-1}$ and $\lambda \in (-1, 1)$. Let $\rho \in \mathcal{O}_m$ so that $\rho(\varepsilon) = \varepsilon$. If W is a q -design and W'_λ is a q_1 -design, then $W_\rho = (W \setminus W_\lambda) \cup \rho(W_\lambda)$ is a $q \wedge q_1$ -design.*

Proof. It suffices to adapt the proof of Theorem 3.3. \square

In Corollary 3.5, we shall specialize the pole ε . We shall assume that one of its components equals 1. We shall write W''_λ to denote the subset of S^{m-2} obtained

from W'_λ , when we ignore the zero components of its elements which correspond to the nonzero entry of ε .

Corollary 3.5. *Let W be a finite nonempty subset of S^{m-1} , $\lambda \in (-1, 1)$ e $\rho \in \mathcal{O}_m$. Let ε be as in the previous paragraph. If W is a q -design and W'_λ is a q_1 -design in S^{m-2} , then $W_\rho = (W \setminus W_\lambda) \cup \rho(W_\lambda)$ is a $q \wedge q_1$ -design.*

Proof. It will be omitted. □

Let W_2 be the set considered in the previous section and $\varepsilon = (0, 0, 1) \in \mathbb{R}^3$. The rotation $\rho \in \mathcal{O}_3$ defined by

$$\rho(x, y, z) = \begin{pmatrix} \cos \theta & \sin \theta & 0 \\ -\sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix},$$

satisfies $\rho(\varepsilon) = \varepsilon$. If $\lambda = 0$ then $W_\lambda = W'_\lambda = \{(1, 0, 0), (0, 1, 0), (-1, 0, 0), (0, -1, 0)\}$ and $W''_\lambda = \{(1, 0), (0, 1), (-1, 0), (0, -1)\}$. Since W''_λ coincides with the design W_1 analyzed in the previous section, W''_λ is a 3-design in S^1 . A procedure similar to that used in Section 2 shows that W is a 3-design in S^2 . Hence, Corollary 3.5 is applicable and in the case $\theta = \pi/4$ we can conclude that the union of

$$W \setminus W_\lambda = \{(0, 0, 1), (0, 0, -1)\}$$

and

$$\rho(W_\lambda) = \left\{ \left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}, 0 \right), \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, 0 \right), \left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, 0 \right), \left(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}, 0 \right) \right\}$$

is a 3-design in S^2 .

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