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Packing Polyhedra within Convex Sets

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ABSTRACT. This work addresses a problem related to the packing of identical regular polyhedra in arbitrary Euclidean space. The problem aims to maximize the number of identical regular polyhedra that can be packed within a convex set, referred to as the domain. The problem is modeled as a nonlinear differentiable programming problem.

Keywords: packing polyhedra, separating hyperplanes, non-linear programming.

1 INTRODUCTION

The problem of packing items within bounded domains in Euclidean space has numerous applications in computer graphics, physics, engineering, and the arts. These problems exhibit varying levels of complexity. Analytical solutions exist for some special cases when the domain is unbounded [7, 10]. Generally, the primary challenge in packing problems within bounded domains is to determine the densest non-lattice arrangement of equal, non-overlapping solid items in Euclidean space.

Packing problems in bounded two-dimensional domains are extensively studied due to their numerous applications in various fields, including cutting stock problems, bin packing, and computational geometry. Theoretical and numerical approaches are widely employed to solve these problems, as demonstrated in the following papers [1, 3, 5, 8, 11, 12].

In [16], the authors present a nonlinear mathematical model for a packing problem involving polygons and circles with free rotation, where the container is a rectangular envelope. The objective is to minimize the area of the rectangular envelope. In this model, separating lines are utilized to ensure the non-overlapping placement of the items.

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In [15], quasi phi-functions were employed to analytically express the non-overlapping relations of irregular polygons in two-dimensional problems. The quasi phi-function allows for the reduction of optimization packing problems to nonlinear programming problems. In general, phi functions are non-differentiable.

Detecting immersion and overlap are central challenges in packing problems. Several methods exist to address these challenges for polygons and circles. For immersion, if the items are polygons and the convex container is defined by inequalities, it suffices to verify that the vertices satisfy these inequalities. However, if the items are circles and the container is elliptical, the process becomes more complicated, as described in [6]. Detecting overlap depends on the geometry of the items. It is straightforward if the items are circles, but when the items are polygons, it is necessary to adopt more elaborated strategies, such as line separators [13]. Phi-functions or quasi phi-functions were utilized by [15, 17] to detect overlapping. The phi-functions considered in both papers involve maximum operators, which implies the non-differentiability of these functions.

In this paper, the line separating method is generalized to n-dimensional Euclidean space for packing polytopes and spheres. Two mathematical models for packing polygons are studied in the two-dimensional case. The first model is polynomial, involving only sums and products. The second model is a special case of the first, in which certain constraints are parameterized using trigonometric functions, thereby reducing the number of constraints and variables. Additionally, a heuristic algorithm is proposed to effectively solve a constrained packing problem, in which the goal is to determine the maximum number of identical items that can be packed into a given bounded convex container. An appropriate initial configuration is selected to increase the change of finding a global solution.

The main contributions of this paper can be summarized as follows:

- *Development of a Multidimensional Packing Model*: A closed model for packing n-dimensional polygons within n-dimensional convex sets is introduced.
- *Formulation as Differentiable Nonlinear Programming*: The packing problem of identical polygons is formulated as differentiable nonlinear programming models.
- *Models and Experiments for Fixed-Size Items*: Two mathematical models utilizing dimensional properties are presented, along with the corresponding numerical experiments.

This paper is organized as follows. The packing problem is formally introduced in Section 2. A nonlinear programming model for several packing problems is presented in Section 3. Section 4 discusses a mathematical model for maximizing the density of polygons under isometric transformations within a fixed container, without overlap. Section 5 presents stochastic multi-start strategies for approximating a global solution, along with numerical results for two- and three-dimensional Euclidean spaces. Conclusions are provided in Section 6.

2 PROBLEM DESCRIPTION

Let Ω be a convex, compact set in the Euclidean space, with volume $v(\Omega) < \infty$, where the volume $v(\Omega)$ is given by the integral

$$v(\Omega) = \int_{\Omega} 1.$$

Definition 2.1. *A packing on a set Ω is a collection $\mathscr{P}$ of subsets of Ω having the following properties:*

1. $v(P) > 0$ *for every* $P \in \mathscr{P}$,
2. $v(P \cap Q) = 0$ *for any distinct* $P, Q \in \mathscr{P}$.

An element P of the collection $\mathscr{P}$ is referred to as an *item*, and Ω is called the *container* or *domain.*

Let P and Q subsets of the Euclidean space $\mathbb{R}^n$. Then P and Q are said to be *congruent* if there exists an isometry, that is, a bijective distance preserving map $T : \mathbb{R}^n \to \mathbb{R}^n$ that transforms one set into the other.

Problem 2.1. *Let $\Omega \subset \mathbb{R}^n$ a bounded convex domain and $\overline{\mathscr{P}} \subset \mathbb{R}^n$ a ser of polyhedra. For every $P \in \overline{\mathscr{P}}$, determine an isometry $T_P : \mathbb{R}^n \to \mathbb{R}^n$ such that the set $\mathscr{P} = \{T_P(P) \,|\, P \in \overline{\mathscr{P}}\}$ is a packing on Ω.*

Problem 2.1 is known as the Identical Item Packing Problem (IIPP) if the items in $\mathscr{P}$ are congruent. This means the emphasis is on finding an arrangement that allows the maximum number of small items to be accommodated. This concept is discussed in [20].

Problem 2.1 is also known as the Cutting and Packing (C&P) problem, and it falls into the category of NP-hard problems. This means that there is no known polynomial-time algorithm to solve these problems in general, even if the items are cubes [9]. This category includes problems like the bin packing problem and sphere packing.

If $\mathscr{P}$ is a finite set of polyhedra, the maximum number of elements in $\mathscr{P}$ that can be placed within a bounded convex domain Ω can be estimated by comparing their total volume. If the combined volume of all the polyhedra in $\mathscr{P}$ exceeds the volume of Ω, then it is impossible to place all the polyhedra within Ω without overlapping.

The *density* d of a packing $\mathscr{P}$ in a domain Ω is defined as the ratio of the total volume of the packed polyhedra to the volume of Ω. Consequently, $d \leq 1$ and

$$d = \frac{1}{v(\Omega)} \sum_{P \in \mathscr{P}} v(P). \tag{2.1}$$

If all elements of the packing $\mathscr{P}$ are congruent and have exactly m elements, then they all have the same volume. Therefore, the total volume of the packing is $m\,v(P)$

$$d = \frac{m\,v(P)}{v(\Omega)}, \tag{2.2}$$

where $v(P)$ is the volume of each element.

Considering that each item has a non-zero volume and the container has a finite volume, there is an upper bound on the number of items that can fit within the container in the packing $\mathscr{P}$. The presence of an upper bound is a key aspect in optimization and packing problems, as the main challenge is to find the maximum capacity of the container or to minimize the losses in Cutting and Packing (C&P) problems.

$$0 \leq m \leq \frac{v(\Omega)}{v(P)}, \tag{2.3}$$

where m is the number of identical items. Finding the optimal cutting and packing configuration is equivalent to maximizing the value of m, which represents the number of items that can be placed within the domain without any overlap.

Problem 2.2. *Let Ω be a bounded convex container and P a polyhedron. Let's denote by $\mathscr{P}_m$ a set formed by m copies of P. Find the greatest value of $m \geq 0$ such that $\mathscr{P}_m$ is a packing in Ω.*

The problem 2.2 is well-defined by inequality (2.3), which ensures the existence of a maximum number of packings under the given constraints.

The problem becomes more complex when the packing involves items of varying shapes. Moreover, for a fixed number of items, the problem of minimizing the volume of the container is also well-defined. However, this paper focuses on the scenario in which the container is fixed.

3 NONLINEAR PROGRAMMING MODELS

This section introduces a differentiable nonlinear programming model for solving problems 2.1 and 2.2 in $n-$dimensional spaces.

Lemma 3.1. *Let P and Q be two compact sets in $\mathbb{R}^n$, each with finite, positive volume $v(P) > 0$ and $v(Q) > 0$. Then:*

$$v(P \cap Q) = 0 \textit{ if and only if } \operatorname{int}(P) \cap \operatorname{int}(Q) = \emptyset. \tag{3.1}$$

Proof. If $\operatorname{int}(P) \cap \operatorname{int}(Q) \neq \emptyset$, then by definition there exist a non-empty open ball $B \subset \operatorname{int}(P) \cap \operatorname{int}(Q)$, with positive volume, i.e. $v(B) > 0$. Since $B \subset \operatorname{int}(P) \cap \operatorname{int}(Q) \subset P \cap Q$, it follows that $v(P \cap Q) \geq v(B) > 0$, which implies $v(P \cap Q) \neq 0$. This proves the contrapositive of the forward direction.

Suppose, conversely that $\operatorname{int}(P) \cap \operatorname{int}(Q) = \emptyset$, then $P \cap Q \subset partial P \cap \partial Q$. Indeed, if $x \in P \cap Q$, then $x \notin \operatorname{int}(P \cap Q) = \operatorname{int}(P) \cap \operatorname{int}(Q)$, which is empty by hypothesis. Thus $x \notin \operatorname{int}(P)$ and $x \notin$

int(Q), which implies that $x \in \partial P$ and $x \in \partial Q$. On the other hand, since P and Q has finite volume and are compact, their boundaries have measure zero [18]. Therefore $v(P \cap Q) \leq v(\partial P) + v(\partial Q) = 0 + 0.$, which implies $v(P \cap Q) = 0$. □

Suppose P and Q are two convex sets that do not overlap; that is, the volume of their intersection is zero. By Lemma 3.1 $\text{int}(P) \cap \text{int}(Q) = \emptyset$, then by the geometric form of the Hahn-Banach theorem [4], there exist a nonzero vector $a \in \mathbb{R}^n$ and a scalar b such that $\langle a, x \rangle \leq b$ for all $x \in P$ and $\langle a, x \rangle \geq b$ for all $x \in Q$. In other words, the affine function $\langle a, x \rangle - b$ is non-positive on P and non-negative in Q, the hyperplane is called a *separating hyperplane* for the sets P and Q.

A bounded polyhedron is determined by its set of extreme points, also known as vertices, which form a finite subset of points in Euclidean space, or equivalently, it can be described as the intersection of a finite number of half-spaces [2]. Let P be a polyhedron with V vertices, therefore, there exists a finite set of extreme points $\{p^1, \ldots, p^V\} \subset \mathbb{R}^n$ such that

$$P = \text{conv}\{p^1, \ldots, p^V\},$$

where conv denotes the convex hull of the set os points $\{p^1, \ldots, p^V\}$.

To determine whether two polyhedra overlap, it is sufficient to check whether there exists a hyperplane that separates the sets of extreme points of the polyhedra.

Let P^i and P^j be two polyhedra in the collection $\mathscr{P}_m$ with their extreme points $p^{i,1}, \ldots, p^{i,V_i}$ and $p^{j,1}, \ldots, p^{j,V_j}$ respectively. If the polyhedra P^i and P^j do not overlap, then there exists a non-zero vector $a_{i,j} \in \mathbb{R}^n$ and a scalar $b_{i,j} \in \mathbb{R}$ such that the following conditions hold: for all $i = 1, \ldots, m$

$$\langle a_{i,j}, p^{i,s} \rangle \leq b_{i,j} \quad \text{for all } s = 1, \ldots, V_i, \tag{3.2}$$

$$\langle a_{i,j}, p^{j,t} \rangle \geq b_{i,j} \quad \text{for all } t = 1, \ldots, V_j, \tag{3.3}$$

this vector $a_{i,j}$ and scalar $b_{i,j}$ define a separating hyperplane.

The hyperplane $\{x \in \mathbb{R}^n \mid \langle a_{i,j}, x \rangle = b_{i,j}\}$ effectively separates the interiors of the two polyhedra P^i and P^j, ensuring that no points from the interior of P^i lie on the same side of the hyperplane as any points from the interior of P^j. The existence of such hyperplanes guarantees that any two elements in $\mathscr{P}$ do not overlap.

Let Ω be a convex domain described by a finite number of algebraic inequalities. Then, verifying that the vertices of a polyhedron satisfy these inequalities is sufficient to confirm that the entire polyhedron lies within the domain, because Ω is a convex set. If the container is defined by a system of inequalities such as:

$$\Omega = \{x \in \mathbb{R}^n \mid g_i(x) \leq 0 \text{ for all } i = 1, \ldots, M\}, \tag{3.4}$$

A the polyhedron is contained within the convex set Ω if, and only if, all its vertices are in the container. Equivalently, for all $i = 1, \ldots, m$, the following holds:

$$p^{i,s} \in \Omega, \quad \forall s = 1, \ldots, V_i, \Leftrightarrow P^i \subset \Omega, \tag{3.5}$$

where P^i denotes the polyhedron associated with the set of extreme points $p^{i,s}$.

The primary issue with equations (3.2) and (3.3) is ensuring the non-zero condition for the vector $a^{i,j}$. This can be addressed by imposing additional conditions, such as:

$$\|a^{i,j}\|^2 = 1, \tag{3.6}$$

for every P^i and P^j. Figure 1 illustrates the two-dimensional application of separating hyperplanes for polyhedral packing.

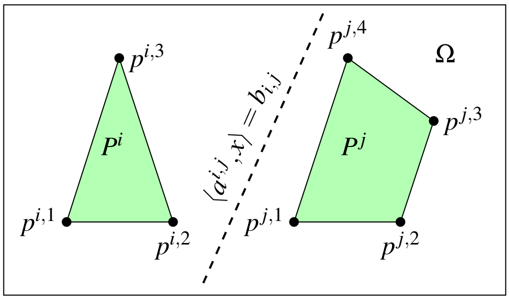


Figure 1: Illustration of the separating hyperplane.

There are numerous methods to solve Problem 2.1. Let $\overline{\mathscr{P}} = \{P_0^1, \dots, P_0^m\}$ be the given set of polyhedra. A crucial step involves determining the displacement of each polyhedron, which necessitates identifying the isometric transformation that turns $P_0^i \in \overline{\mathscr{P}}$ into $P^i \in \mathscr{P}$ for $i = 1, \dots, m$.

An isometric transformation $T_i : \mathbb{R}^n \to \mathbb{R}^n$ is defined by a translation vector c^i and an orthogonal matrix R^i as follows:

$$T_i(x) = c^i + R_i(x).$$

The transformed polygon $T_i(P_0^i) = P^i$ represents the packing configuration within the domain Ω. The vertices $p^{i,s}$ of the polyhedron P^i are derived from the vertices $p_0^{i,s}$ of P_0^i:

$$p^{i,s} = c^i + R_i(p_0^{i,s}). \tag{3.7}$$

To solve Problem 2.2, Problem 2.1 is solved iteratively until the maximum feasible value m^* is determined. To decide if it is possible to pack m items, the constrained problem.

$$
\begin{aligned}
R_i^T R_i &= I_n, && \text{for all } i = 1, \dots, m, && (3.8)\\
\langle a^{i,j}, p^{i,s}\rangle &\le b_{i,j}, && \text{for all } s = 1, \dots, V_i, && (3.9)\\
\langle a^{i,j}, p^{j,t}\rangle &\ge b_{i,j}, && \text{for all } t = 1, \dots, V_j, && (3.10)\\
\|a^{i,j}\|^2 &= 1, && i, j \in \{1, \dots, m\}, i < j, && (3.11)\\
p^{i,s} &\in \Omega, && \text{for all } s = 1, \dots, V_i, && (3.12)
\end{aligned}
$$

must be solved.

In (3.8-3.12), the variables $c^i \in \mathbb{R}, R_i \in \mathbb{R}(n,n)$ represent the $i-$th translation and rotation, respectively. The vertex coordinates of $p^{i,s}$ can be recovered using equation (3.7). This model is referred to as the Polynomial Model (PM) because it is defined by polynomial operations.

The unknowns of the nonlinear feasibility problem (3.8-3.12) are $c^i, R_i, a^{i,j}, b_{i,j}$ $i, j \in \{1,\ldots,m\}$ with $i < j$. The equalities and inequalities in (3.8-3.11) define $n^2m + \frac{1}{2}m(m+1)(2V+1)$ nonlinear constraints, where all polygon are congruent with V vertices. Additionally, (3.12) define mVL constraints, where L are the number of constraints that define Ω.

To pack polygons and spheres, one of the constraints (3.10) or (3.9) can be replaced with the constraints

$$(\langle a^{i,j}, c^i \rangle - b_{i,j})^2 \geq r^2 \|a^{i,j}\|^2, \qquad i, j \in \{1,\ldots,m\}, i < j \tag{3.13}$$

$$\langle a^{i,j}, c^i \rangle - b_{i,j} \leq 0, \qquad i, j \in \{1,\ldots,m\}, i < j, \tag{3.14}$$

where $c^i \in \mathbb{R}^n$ is the center of the sphere and $r > 0$ is the radius of the sphere. The constraints (3.13) guarantees that the distance to the hyperplane separator must be greater than the radius of the sphere being considered, and (3.14) ensures it is on the opposite side of the other item. However, in this paper, only congruent or identical items are considered.

3.1 Model based on trigonometric functions

The specific properties of the two-dimensional space inherent to the Polynomial Model (PM) are investigated and simplified through the parametrization of the rotation matrix and the normal vector of the hyperplane separator. This simplification leads to a reduction in the number of variables and constraints. However, the incorporation of trigonometric functions within the model can potentially increase computational demands. This new model is termed the Trigonometric Model (TM).

Regular polygons P_0^i for $i = 1,\ldots,m$ are generated with vertices defined as follows:

$$p_0^{i,k} = \begin{bmatrix} r\cos(\frac{2\pi k}{V}) \\ r\sin(\frac{2\pi k}{V}) \end{bmatrix}, \quad \forall k = 1,\ldots,V. \tag{3.15}$$

where V denotes the number of vertices and r represents the circumradius.

Let $c^i = (c_x^i, c_y^i)$ be the circumcenter of the regular polygon P_i, with circumradius $r > 0$ and $i \in \{1,2,\ldots,m\}$. The vertices $p^{i,k} = (p_x^{i,k}, p_y^{i,k})$ of the polygon can be expressed using the center c^i and the rotation of the polygon, represented by a parameter α_i.

$$\begin{bmatrix} p_x^{i,k} \\ p_y^{i,k} \end{bmatrix} = \begin{bmatrix} c_x^i \\ c_y^i \end{bmatrix} + \begin{bmatrix} \cos(\alpha_i) & -\sin(\alpha_i) \\ \sin(\alpha_i) & \cos(\alpha_i) \end{bmatrix} \begin{bmatrix} r\cos(\frac{2\pi k}{V}) \\ r\sin(\frac{2\pi k}{V}) \end{bmatrix}, \quad \forall k = 1,\ldots,V. \tag{3.16}$$

Consequently, we have:

$$p_x^{i,k} = c_x^i + r\cos\left(\frac{2\pi k}{V} + \alpha_i\right), \qquad \forall k = 1,\ldots,V, \tag{3.17}$$

$$p_y^{i,k} = c_y^i + r\sin\left(\frac{2\pi k}{V} + \alpha_i\right), \qquad \forall k = 1,\ldots,V, \tag{3.18}$$

where V represents the number of vertices of the regular polygon.

The normal vector $a^{i,j} \in \mathbb{R}^2$ can be parameterized as follows:

$$a^{i,j} = \begin{bmatrix} \cos(\eta_{i,j}) \\ \sin(\eta_{i,j}) \end{bmatrix}, \tag{3.19}$$

where $\eta_{i,j} \in [0, 2\pi]$ is the angle of the vector $a^{i,j}$. The parametrization of the normal vector (3.19) satisfies the constrain of being non-zero, specifically $\|a^{i,j}\| = 1$.

To determine whether two regular polygons $P^i \subset \mathbb{R}^2$ and $P^j \subset \mathbb{R}^2$ with circumradius r in the planar case are non-overlapping, it is necessary to verify the inequalities given by equations (3.2) and (3.3). These inequalities can be expressed as follows for this specific case:

$$\cos(\eta_{i,j})\, p_x^{i,k}(c_x^i, \alpha_i) + \sin(\eta_{i,j})\, p_y^{i,k}(c_y^i, \alpha_i) \le b_{i,j}, \qquad \forall k = 1,\ldots,V. \tag{3.20}$$

$$\cos(\eta_{i,j})\, p_x^{j,k}(c_x^j, \alpha_j) + \sin(\eta_{i,j})\, p_y^{j,k}(c_y^j, \alpha_j) \ge b_{i,j}. \qquad \forall k = 1,\ldots,V. \tag{3.21}$$

Finally, we choose a circumference as our arbitrary container Ω; however, the domain or container could be any convex set described by a finite number of algebraic constraints.

$$\Omega = \left\{(x,y) \in \mathbb{R}^2; x^2 + y^2 < \rho^2\right\}. \tag{3.22}$$

To guarantee that the polygon P^i is entirely contained within Ω, we require that, for each convex $k = 1,\ldots,V$ of polygon P^i, the following inequality holds:

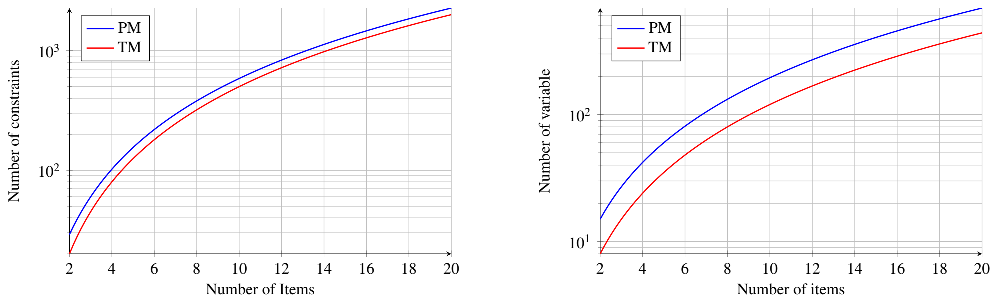
$$(p_x^{i,k})^2 + (p_y^{i,k})^2 \le \rho^2, \tag{3.23}$$

where ρ is the radius of the circumference, and $i = 1,\ldots,m$ items.

Constraints (3.20), (3.21), and (3.23) define the feasibility problem of packing m polygons within a circular domain. The upper bound for this packing problem is given as follows:

$$0 \le m \le \frac{\pi\rho^2}{\frac{V}{2} r^2 \sin(\frac{2\pi}{V})}. \tag{3.24}$$

In Figure 2, the growth in the number of constraints is shown for both the TM and PM models as a function of the number of pentagonal items. Figure 2a illustrates that the PM model requires more constraints than the TM model. Additionally, Figure 2b shows that the PM model also involves a greater number of variables compared to the TM model.



(a) Number of inequality constraints in the constrained packing problem for pentagons.

(b) Number of variables in the constrained packing problem for pentagons.

Figure 2: Evolution of the number of constraints and variables.

4 MAXIMIZING POLYGON DENSITY IN A CONVEX SET

This section explores the problem of packing congruent polygons within larger domains, aiming to maximize the packing density. Given a total of m congruent polygons, our objective is to arrange all m regular polygons without overlaps inside larger domains while maximizing the size of the polygons. In other words, we want to maximize the common circumradius of the polygons.

We denote the maximum achievable circumradius by r, and the arrangement that achieves this maximum circumradius is referred to as an optimal packing.

$$\text{Maximize} \quad r \tag{4.1}$$

subject to:

$$\begin{aligned}
\cos(\eta_{i,j})\, p_x^{i,k} + \sin(\eta_{i,j})\, p_y^{i,k} &\le d_{i,j}, && \forall i,j \in \{1,\ldots,m\},\ i<j, k=1,\ldots,V, && (4.2)\\
\cos(\eta_{i,j})\, p_x^{j,k} + \sin(\eta_{i,j})\, p_x^{j,k} &\ge d_{i,j}, && \forall i,j \in \{1,\ldots,m\},\ i<j, k=1,\ldots,V, && (4.3)\\
(p_x^{i,k})^2 + (p_y^{i,k})^2 &\le \rho^2 && k=1,\ldots,V,\ i=1,\ldots,m, && (4.4)\\
r &\ge 0. && && (4.5)
\end{aligned}$$

Note that r is a variable in this problem, unlike the previews instances where it was given constant. The relationship is given by $p^{i,s} = c^i + R(\alpha_i) p_0^{i,s}(r)$ where $p_0^{i,s}(r)$ is defined as:

$$p_0^{i,k}(r) = \begin{bmatrix} r\cos(\frac{2\pi k}{V}) \\ r\sin(\frac{2\pi k}{V}) \end{bmatrix}, \quad \forall k = 1,\ldots,V, \tag{4.6}$$

that is, $p^{i,s} = p^{i,s}(c^i, \alpha_i, r)$.

5 NUMERICAL EXPERIMENTS

This section presents the results of numerical experiments conducted for Problem 2.2, which involves packing regular polygons, each with a fixed circumradius r, using the Polynomial Model and the Trigonometric Model . All experiments were performed on a 4 GHz AMD A10-5800B APU processor with 4 GB of RAM, running the GNU/Linux operating system (Linux Mint 21, kernel version 5.15.0-124-generic). The implementation was carried out using JuMP [14], a modeling language for mathematical optimization within the Julia programming environment.

We chose IPOPT (Interior Point OPTimizer) [19] as the local solver. IPOPT is a software package designed for large-scale nonlinear optimization. It is designed to find local solutions to the mathematical optimization problem

$$\min_{x\in\mathbb{R}^n} \quad f(x) \tag{5.1}$$

$$\text{subject to} \quad g_L \leq g(x) \leq g_U, \tag{5.2}$$
$$x_L \leq x \leq x_U,$$

where $f:\mathbb{R}^n \to \mathbb{R}$ is the objective function, and $g:\mathbb{R}^n \to \mathbb{R}^m$ is the constraint functions. The vectors g_L and g_U denote the lower and upper bounds on the constraints, and the vectors x_L and x_U are the bounds on the variables x. The functions $f(x)$ and $g(x)$ can be nonlinear and nonconvex, but should be twice continuously differentiable.

Our objective is to maximize the number of regular polygons with circumradius r that can be packed within a given circumference. If, for a specified integer m the feasibility problem PM or TM admits a solution, then the maximal number of polygons that can be packed is at least m.

The optimization problem is solved through a feasibility-based iterative process. Given a lower bound m_{lb} on the number of congruent polygons, the procedure begins by attempting to solve a feasibility problem with $m = m_{lb}+1$. If a feasible solution is found, we set $m_{lb} \leftarrow m$ and try again. This iteration proceeds until a value of m is reached for which no feasible solution to the equations can be found. At this point, the maximum packing solution is given by $m^* = m-1$. Similar methods have been employed in [6] and [3]. If the problem (5.1)- (5.2) with $f \equiv 0$, admits a feasible solution for a given m, the value of m is incremented by one and the process repeated. Otherwise, the same value of m is retained, and attempts continue until an imposed CPU time limit is reached. The whole procedure is described by Algorithm 1.

A similar algorithm to Algorithm 1 can be formulated to solve Problem 2.2 using the Model (PM). The main difference are in Steps 5, 6 and 7, for solving the model (PM) the initial guess is specified as:

$$x, y, a_x, a_y, b, R. \tag{5.3}$$

These variables represent the initial estimates for the decision variables required to initialize the optimization procedure under the Polynomial Model framework. In the following subsections we describe each step of Algorithm 1 in detail.

Algorithm 1 Polygon Packing Algorithm (TM)

1: Given a domain Ω.
2: Compute a lower bound $m_{lb} \geq 0$ such that $x, y, \alpha \in \mathbb{R}^{m_{lb}}, \eta, b \in \mathbb{R}^{m_{lb}(m_{lb}-1)/2}$ are feasible (note that $m_{lb} = 0$ is a valid choice).
3: Set $m \leftarrow m_{lb} + 1$.
4: **while** time limit is not achieved **do**
5: Compute an initial guess (x, y, α, η, b) for problem (5.1)-(5.2).
6: Find a stationary point $(x^*, y^*, \alpha^*, \eta^*, b^*)$ of the feasibility problem given by TM, starting from x, y, α, η, b.
7: **if** $(x^*, y^*, \alpha^*, \eta^*, b^*)$ is feasible **then**
8: Set $m \leftarrow m + 1$.
9: **end if**
10: **end while**
11: Stop and declare $m^* = m_{lb} - 1$ items were packed.

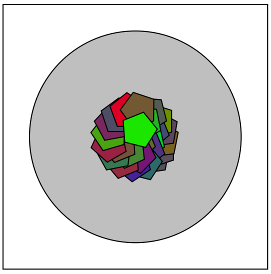
5.1 Strategies for Multistart Initialization

This subsection describes various methods for generating initial guesses to locate a stationary point of the TM and PM model at Step 5 of Algorithm 1. The three types of random initial guesses considered, differ on the domain within which polygon variable are generated. Type G_1 corresponds to random values $x, y \in [-\varepsilon r, \varepsilon r]$, $\alpha, \eta \in [0, 2\pi]$ and $b \in [-\frac{1}{4}r, \frac{1}{4}r]$. Type G_2 corresponds to random values $(x, y) \in [x_L, x_U] \times [y_L, y_U]$, $\alpha, \eta \in [0, 2\pi]$ and $b \in [-\frac{1}{4}r, \frac{1}{4}r]$. The type G_3, the initial guess to pack m polygons consists of the known solution for packing $m-1$ polygons, plus a random initial value generated as in strategy G_1 for the five unknown variables with the additional polygon. These three types of random initial guesses are depicted in Figure 3.

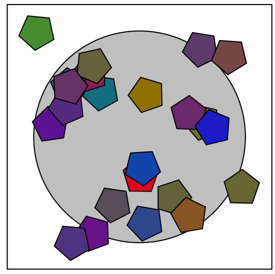
The impact of three different initial guess types on Algorithm 1 was assessed. The lower bound m_{lb} was set to zero, and the CPU time was limited to 10 minutes. When employing initial guess types G_1, G_2 and G_3, Algorithm 1 performed faster with G_3, compared to G_1 and G_2. However good results were obtained only for G_2 in terms of maximizing of packed items, while G_3 in speed. The least effective initial guess type was determined by G_1. As described in the Figure 4.

After observing the results obtained by G_1, G_2 and G_3, a new strategy, G_4 was defined. The first trial of this strategy is referred to G_3. However, if an infeasible packing arrangement is found for m polygons, all variables are multiplied by an random scalar ε, generally chosen as $\frac{1}{4}$. This process generates a configuration similar to G_2. This modified configuration serves as the initial guess for the next trial.

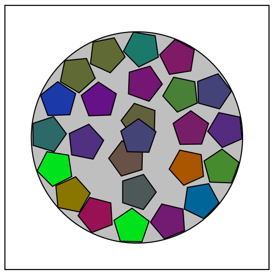
In the first set of experiments, we fixed the time limit (T = 10 minutes) and solved instances with polygons ranging from equilateral triangles to hendecagons, inscribed within a sphere of radius $\rho = 4$. Tables 1 and 2 show the numerical solutions found by each method for each polygon, respectively.



(a) Strategy G_1.



(b) Strategy G_2.



(c) Strategy G_3.

Figure 3: Random initial guesses G_1, G_2 and G_3 respectively.

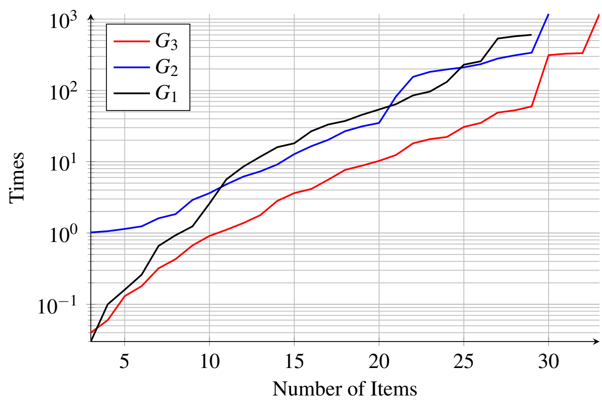


Figure 4: The time evolution concerning the number of items packed by G_1, G_2 and G_3 using the Trigonometric Model.

When the polygon is an equilateral triangle, the solutions obtained by TM and PM differ. TM 61 items, while PM packed 43. Additionally, TM required more computational time than PM.

It is possible to observe that the density found by TM is higher than that found by PM. However, the computational time required to solve the model TM is greater than that of PM, as observed in Figure 7.

Table 1: Numerical results obtained by applying Algorithm 1 combined with initial guesses of type G_4 using the Trigonometric Model TM.

Polygon	**Items**	**Density**	**Bound**	**Trial**	**Time (s)**
Triangle	61	0.7700	78.97	1	586.3482
Square	38	0.7409	51.29	3	340.6758
Pentagon	32	0.7417	43.14	2	668.2621
Hexagon	30	0.7598	39.48	4	970.2528
Heptagon	28	0.7469	37.49	4	977.4046
Octagon	27	0.7444	36.27	2	480.2960
Nonagon	27	0.7613	35.46	2	346.6350
Decagon	26	0.7449	34.90	2	604.9014
Hendecagon	25	0.7247	34.50	1	587.2593

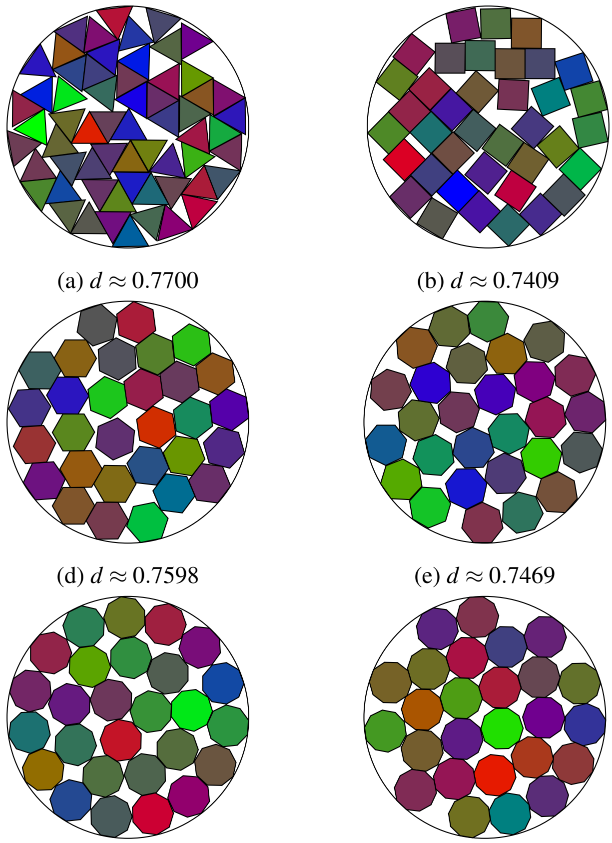
Table 2: Numerical results obtained by applying Algorithm 1 combined with initial guesses of type G_4 using the Polynomial Model PM.

Polygon	**Items**	**Density**	**Bound**	**Trial**	**Time (s)**
Triangle	43	0.5445	78.97	0	5772.3890
Square	38	0.7409	51.29	0	121.6695
Pentagon	33	0.7649	43.14	0	297.1452
Hexagon	29	0.7345	39.48	0	80.6331
Heptagon	23	0.6135	37.49	0	820.3773
Octagon	27	0.7444	36.27	0	308.9169
Nonagon	27	0.7613	35.46	0	330.0430
Decagon	27	0.7735	34.90	0	158.4462
Hendecagon	26	0.7537	34.50	0	143.9524

5.2 Spatial Experiments

Finally, the three-dimensional packing problem is considered using the polynomial model, as it is straightforward to implement once the vertices of the three-dimensional polyhedron are determined.

For numerical purpose the polyhedron to be packed were chosen to be icosahedron. An icosahedron is a polyhedron characterized by 20 equilateral triangular faces, 30 edges, and 12 vertices. It is one of the five Platonic solids, known for its high degree of symmetry. Each vertex of an icosahedron is shared by five triangular faces, and the structure exhibits icosahedral symmetry, meaning that it can be rotated in space without altering its appearance. The icosahedron plays an important role in geometry, crystallography, and various models in physics, particularly in the study of spherical symmetry.



(a) $d \approx 0.7700$ (b) $d \approx 0.7409$ (c) $d \approx 0.7417$

(d) $d \approx 0.7598$ (e) $d \approx 0.7469$ (f) $d \approx 0.7444$

(g) $d \approx 0.7613$ (h) $d \approx 0.7449$ (i) $d \approx 0.7247$

Figure 5: Graphical representation of the solution obtained by strategy G_4 using the trigonomtric model TM.

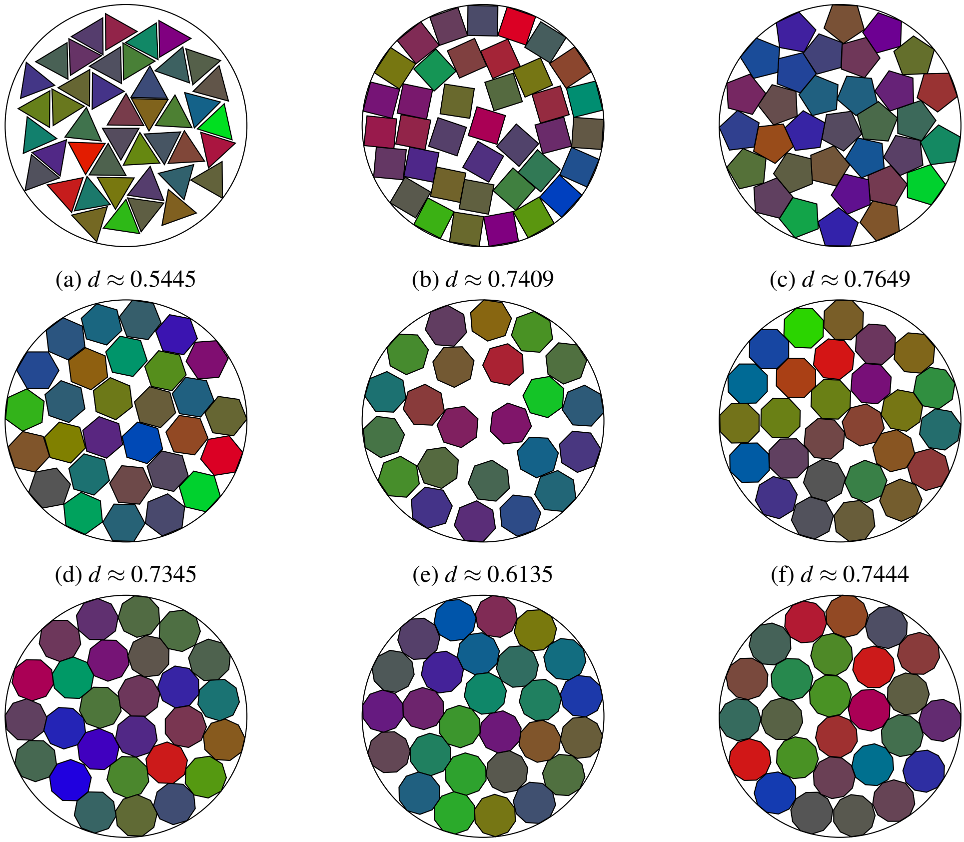
A possible Cartesian coordinate representation for the vertices of a regular icosahedron with an edge length of two is given by:

$$\begin{bmatrix} 0 \\ \pm 1 \\ \pm\phi \end{bmatrix}, \quad \begin{bmatrix} \pm 1 \\ \pm\phi \\ 0 \end{bmatrix}, \quad \begin{bmatrix} \pm\phi \\ 0 \\ \pm 1 \end{bmatrix}. \tag{5.4}$$

where $\phi = \frac{1+\sqrt{5}}{2}$ denotes the golden ratio.

For $i \in \{1, 2, \ldots, m\}$, a vertex of the icosahedron P^i in the space can be expressed as

$$p^{i,s} = c^i + R_i V_r^s \qquad \forall s = 1, \ldots, 12. \tag{5.5}$$



(a) $d \approx 0.5445$ (b) $d \approx 0.7409$ (c) $d \approx 0.7649$

(d) $d \approx 0.7345$ (e) $d \approx 0.6135$ (f) $d \approx 0.7444$

(g) $d \approx 0.7613$ (h) $d \approx 0.7735$ (i) $d \approx 0.7537$

Figure 6: Graphical representation of the solution obtained by strategy G_4 using the polynomial model PM.

where V_r^s denote the s-th column of the matrix

$$V_r = r \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 1 & -1 & -1 & \phi & \phi & -\phi & -\phi \\ 1 & 1 & -1 & -1 & \phi & -\phi & \phi & -\phi & 0 & 0 & 0 & 0 \\ \phi & -\phi & \phi & -\phi & 0 & 0 & 0 & 0 & 1 & -1 & 1 & -1 \end{bmatrix}. \tag{5.6}$$

Finally, to decide if it is possible to pack m identical icosahedron is necessary to solve the constrain problem

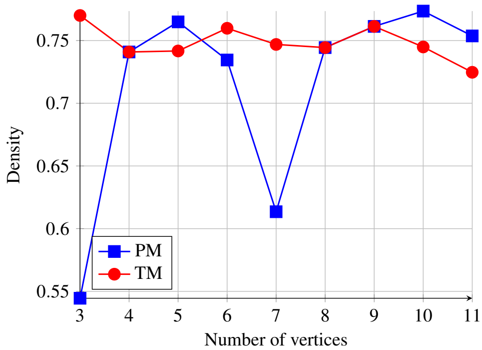
$$R_i^T R_i = I_3, \qquad \forall i \in \{1\ldots,m\}, \tag{5.7}$$

$$\langle a^{i,j}, p^{i,s}\rangle \le b_{i,j}, \qquad \forall i,j \in \{1\ldots,m\}, i<j, s=1,\ldots,12, \tag{5.8}$$

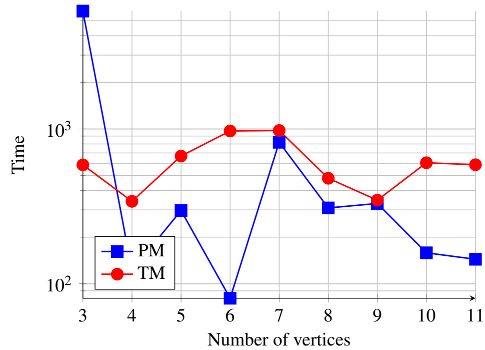
$$\langle a^{i,j}, p^{j,t}\rangle \ge b_{i,j}, \qquad \forall i,j \in \{1\ldots,m\}, i<j, t=1,\ldots,12, \tag{5.9}$$

$$\|a^{i,j}\|^2 = 1, \qquad \forall i,j \in \{1\ldots,m\}, i<j, \tag{5.10}$$

$$\|p^{i,s}\|_2^2 \le \rho^2, \qquad \forall i \in \{1\ldots,m\}, s=1,\ldots,12. \tag{5.11}$$



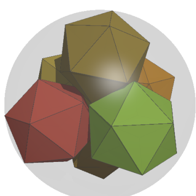
(a) Achieved density



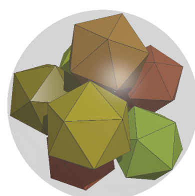
(b) Time used to pack items

Figure 7: Graphical representation of the density and the time used to solve it.

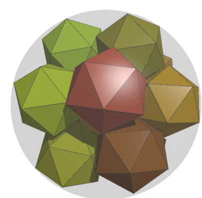
Table 3 summarizes these results, while the figures in Figure 8 illustrate the corresponding solutions.



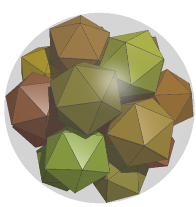
(a) $d \approx 0.3906$



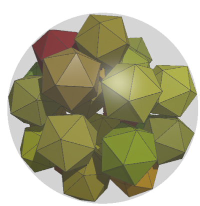
(b) $d \approx 0.3797$



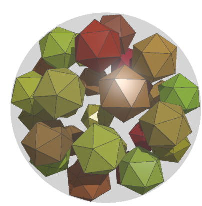
(c) $d \approx 0.4667$



(d) $d \approx 0.4690$



(e) $d \approx 0.3938$



(f) $d \approx 0.2523$

Figure 8: Graphical representation of the solutions obtained by strategy G_4

Figure	Number of items	Edge	Density	Bound	Number of trials	CPU time (s)
8a	6	2.00	0.3906	15.36	0	4.1840
8b	8	1.80	0.3797	21.07	0	7.6161
8c	14	1.60	0.4667	30.00	0	98.2319
8d	21	1.40	0.4690	44.78	0	288.8084
8e	28	1.20	0.3938	71.11	0	619.2771
8f	31	1.00	0.2523	122.88	0	827.8501

Table 3: Effort measurement for the packing icosahedron problem within a sphere using Algorithm 1 and the Polynomial Model (PM).

6 CONCLUSION

This work addressed two mathematical models for packing identical items within a convex set defined by algebraic constraints. The objective was to maximize the number of polygons packed into the container. The proposed models were applicable to items in an n-dimensional Euclidean space. The trigonometric model reduced the number of variables and constraints but required more computational time. In contrast, the polynomial model was faster but tended to produce infeasible solutions.

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Data availability

The complete source codes of the algorithms and models presented in this paper are available at: `https://github.com/hector-fc/packpoly`.

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