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On the Continuous-Time Complementarity Problem

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ABSTRACT. This work deals with solving continuous-time nonlinear complementarity problems defined on two types of nonempty closed convex cones: a polyhedral cone (positive octant) and a second-order cone. Theoretical results that establish a relationship between such problems and the variational inequalities problem are presented. We show that global minimizers of an unconstrained continuous-time programming problem are solutions to the continuous-time nonlinear complementarity problem. Moreover, a relation is set up so that a stationary point of an unconstrained continuous-time programming problem, in which the objective function involves the Fischer-Burmeister function, is a solution for the continuous-time complementarity problem. To guarantee the validity of the K.K.T. conditions for some auxiliary continuous-time problems which appear during the theoretical development, we use the linear independence constraint qualification. These constraint qualification are posed in the continuous-time context and appeared in the literature recently. In order to exemplify the developed theory, some simple examples are presented throughout the text.

Keywords: continuous-time complementarity problem, variational inequalities problem, continuous-time programming problems.

1 INTRODUCTION

Complementarity problems were firstly proposed as the question of finding an $n-$vector x which satisfies the system of inequalities

$$x \geq 0, \qquad Mx+b \geq 0 \qquad \text{and} \qquad x^\top(Mx+b)=0, \tag{1.1}$$

where M is an $n \times n$ matrix, b is an $n-$vector of real numbers and "$\top$" denotes the transposition of vectors and matrices. Such problems are elegant generalizations of certain linear programming, quadratic programming, and game theory problems.

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The importance of problem (1.1) lies in the fact that its form includes several problems by appropriate choices of the vector b and the matrix M. As examples of applications, we can cite the problem of the existence of solutions to linear programs (Cottle [5], Dorn [7]) that can be reduced to a problem in the formulation (1.1), the equilibrium point problem of bimatrix games (Lemke [14]) and the unloading problem for plane curves (Du Val [8]). For other examples of applications, see Isac [11].

The formulation (1.1) was expanded to include a broader class of problems such as nonlinear programming and was rewritten as the problem of finding an $n-$vector x which satisfies the system of inequalities

$$x \geq 0, \qquad f(x) \geq 0 \qquad \text{and} \qquad x^\top f(x) = 0, \tag{1.2}$$

where f is a mapping of $\mathbb{R}^n$ into itself. Among other authors who studied the formulation (1.2), Cottle [5] gave sufficient conditions for the existence of x, and Karamardian [12] established sufficient conditions for the existence of an unique solution.

Bodo and Hanson [3] extended the results of Karamardian [12] to the case where x is an essentially bounded measurable function which maps some finite interval into $\mathbb{R}^n$. Sufficient conditions for the existence of a unique solution are given and applications to continuous linear and nonlinear programming are presented. Such a problem will be called here as Continuous-Time Complementarity Problem (CCP).

In [4], Giannessi first introduced Variational Inequalities Problem (VIP) in a finite-dimensional vector space. Since then, the (VIP) has been extensively studied in a general setting by many authors (see, for example, [9, 13, 21]). Zalmai [22] generalized the VIP for the continuous-time case, presented a generalized sufficiency criteria in continuous-time programming under the concept of invexity and used it to study the existence of a solution for the VIP. Differently of Zalmai, in this paper we use the VIP in order to find solutions to the continuous-time complementarity problem, without assuming convexity.

The main contributions of this work are:

- ⋆ To show that CCP and VIP have the same solutions sets;
- ⋆ Reformulation of the CCP as an Auxiliary Optimization Problem (AOP) and apply optimality conditions to solve it;
- ⋆ Analysis of the conditions under which an AOP global optimal solution is a CCP solution;
- ⋆ Analysis of the conditions under which an AOP stationary point is a CCP solution;
- ⋆ For the analysis mentioned above, we considered two types of cones: polyhedral cone ($\mathbb{R}^n_+$) and second-order cone.

We note that Bodo and Hanson defined the variables of the CCP problem in $L_\infty([0,T];\mathbb{R}^n)$ space (Banach space of all Lebesgue-measurable essentially bounded n-dimensional vector functions), while Zalmai worked in $W([0,T];\mathbb{R}^n)$ space (Hilbert space of all absolutely continuous

n-dimensional vector functions). For standardization purposes, we will work in the $L_\infty([0,T];\mathbb{R}^n)$ space.

The paper is organized as follows. Some preliminaries and results from literature related to continuous-time programming problems with equality and inequality constraints are given in Section 2. In Section 3, we define the continuous-time complementarity problem and the variational-type inequalities problem, and we establish the relationship between these problems. In Section 4, we consider the particular case in which the cone is equal to $\mathbb{R}^n_+$ and, using the Fischer-Burmeister function [10], we derive an unconstrained equivalent problem in the sense that a stationary point of the unconstrained equivalent problem is a solution of the continuous-time complementarity problem. Second-order cones are considered in Section 5. An auxiliary problem with inequality constraints is formulated. Assumptions about the constraint set are made in order to guarantee optimality conditions at optimal solutions. Examples are presented. It is worth mentioning that the approach we followed in Sections 4 and 5 is original even in finite dimension: to the best of our knowledge, there is no work in the literature where results from Sections 4 and 5 were obtained through the approach employed here. Final comments are given in Section 6.

2 PRELIMINARIES

We denote by $\|\cdot\|$ the Euclidean norm and given $f:\mathbb{R}^n\times[0,T]\to\mathbb{R}$, we denote by $\nabla f(x,t)$ the gradient of $f(\cdot,t)$ at x. Consider the continuous-time programming problem with equality and inequality constraints

$$\begin{array}{ll} \text{maximize} & P(x)=\displaystyle\int_0^T \phi(x(t),t)\,dt \\ \text{subject to} & h_j(x(t),t)=0 \text{ a.e. in } [0,T],\ j\in J=\{1,\dots,p\}, \\ & g_i(x(t),t)\geq 0 \text{ a.e. in } [0,T],\ i\in I=\{1,\dots,m\}, \\ & x\in L_\infty([0,T];\mathbb{R}^n), \end{array} \tag{2.1}$$

where $\phi:\mathbb{R}^n\times[0,T]\to\mathbb{R}$, $h_j:\mathbb{R}^n\times[0,T]\to\mathbb{R}$, $j\in J$, and $g_i:\mathbb{R}^n\times[0,T]\to\mathbb{R}$, $i\in I$, are given functions. Here for each $t\in[0,T]$, $x(t)\in\mathbb{R}^n$ and all integrals are given in the Lebesgue sense. $L_\infty([0,T];\mathbb{R}^n)$ denotes the Banach space of all Lebesgue-measurable essentially-bounded n-dimensional vector functions defined on the compact interval $[0,T]\subset\mathbb{R}$, with the norm $\|\cdot\|_\infty$ defined by

$$\|x\|_\infty=\max_{1\leq i\leq n}\ \underset{t\in[0,T]}{\text{esssup}}\ |\,x_i(t)\,|\,.$$

The set of feasible solutions for (2.1) is denoted by

$$\Omega_0=\{x\in L_\infty([0,T];\mathbb{R}^n)\mid h_j(x(t),t)=0,\ g_i(x(t),t)\geq 0 \text{ a.e. in } [0,T],\ i\in I,\ j\in J\}.$$

Definition 2.1. *$x^*\in\Omega_0$ is said to be a local optimal solution for (2.1) if there exists $\varepsilon>0$ such that $P(x)\leq P(x^*)$ for all $x\in\Omega_0$ with $x(t)\in x^*(t)+\varepsilon B$ a.e. $t\in[0,T]$, where B denotes the* *open unit ball with center at the origin in $\mathbb{R}^n$. $x^* \in \Omega_0$ is said to be a global optimal solution if $P(x) \leq P(x^*)$ for all $x \in \Omega_0$.*

Let $\varepsilon > 0$ and x^* a local optimal solution for (2.1). We will assume that

(H1) the functions $\phi(\cdot,t)$, $h_j(\cdot,t)$, $j \in J$, and $g_i(\cdot,t)$, $i \in I$, are twice continuously differentiable on $x^*(t) + \varepsilon\bar{B}$ a.e. in $[0,T]$, where $\bar{B}$ denotes the closed unit ball with center at the origin in $\mathbb{R}^n$;

(H2) the functions $\phi(x,\cdot)$, $h_j(x,\cdot)$, $j \in J$, and $g_i(x,\cdot)$, $i \in I$, are Lebesgue measurable for each x, $\phi(x(\cdot),\cdot)$, $h_j(\cdot,\cdot)$, $j \in J$, and $g_i(x(\cdot),\cdot)$, $i \in I$, are essentially bounded in $[0,T]$ for all $x \in L_\infty([0,T];\mathbb{R}^n)$;

(H3) there exist $K_\phi > 0$ and $K_0 > 0$ such that, for a.e. $t \in [0,T]$, we have that

$$\|\nabla\phi(x^*(t),t)\| \leq K_\phi \quad \text{and} \quad \|\nabla[h,g](x^*(t),t)]\| \quad \leq \quad K_0.$$

Karush-Kuhn-Tucker type optimality conditions can be obtained under additional assumptions, for example, the linear independence constraint qualification (LICQ). The hypothesis below is necessary for the application of the Uniform Implicit Function Theorem (see [15] and [6]).

(H4) There exists an increasing function $\bar{\theta} : (0,\infty) \to (0,\infty)$, $\bar{\theta}(s) \downarrow 0$ when $s \downarrow 0$, such that, for all $\tilde{x}, x \in x^*(t) + \varepsilon\bar{B}$,

$$\|\nabla[h,g](\tilde{x},t) - \nabla[h,g](x,t)\| \leq \bar{\theta}(\|\tilde{x} - x\|) \text{ a.e. } t \in [0,T].$$

The following definition refers to the continuous-time case of the linear independence constraint qualification. Note that we define LICQ requiring a "uniform invertibility" for almost every $t \in [0,T]$. This definition is different from the definition used in the finite dimensional case. For more details, see [15].

Definition 2.2. *We say that the constraint qualification (LICQ) is satisfied at $x^* \in \Omega_0$ if there exists $K > 0$ such that*

$$\det\{\Upsilon(t)\Upsilon(t)^\top\} \geq K \text{ a.e. } t \in [0,T],$$

where

$$\Upsilon(t) = \begin{pmatrix} \nabla h(x^*(t),t) & 0 \\ \nabla g(x^*(t),t) & \operatorname{diag}\{-2w_j^*(t)\}_{j\in I} \end{pmatrix}$$

and $w_j^*(t) = \sqrt{g_j(x^*(t),t)}$ a.e. $t \in [0,T]$, $j \in I$.

Proposition 2.1. *(Theorem 4.2 in do Monte and de Oliveira [15]) Assume that (H1)-(H4) and (LICQ) hold at $x^* \in \Omega_0$ and x^* is a local optimal solution for (2.1). Then there exists $(\tilde{u},\ \tilde{v}) \in L_\infty([0,T];\mathbb{R}_p \times \mathbb{R}^m_+)$ such that, for almost every $t \in [0,T]$, $\tilde{v}_i(t)g_i(x^*(t),t) = 0$, $i \in I$, and*

$$\nabla\phi(x^*(t),t) + \sum_{j=1}^{p} \tilde{u}_j(t)\nabla h_j(x^*(t),t) + \sum_{i=1}^{m} \tilde{v}_i(t)\nabla g_i(x^*(t),t) = 0.$$

We will use the Mangasarian-Fromovitz constraint qualification presented in ([6]). For this purpose, let $b > 0$ real number. We will denote by $I_b(t)$ the index set of $b-$active constraints at $x^* \in \Omega_0$, that is, $I_b(t) = \{i \in I \mid 0 \leq g_i(x^*(t),t) \leq b\}$ for almost every $t \in [0,T]$. For all $i \in I$, let us define the function $\delta_i^b : [0,T] \to \mathbb{R}$ as

$$\delta_i^b(t) = \begin{cases} 1, \ i \in I_b(t), \\ 0, \ \text{otherwise.} \end{cases}$$

The next definition refers to continuous-time case of the Mangasarian-Fromovitz constraint qualification. Note that we define MFCQ requiring a "uniform positivity" for almost every $t \in [0,T]$. This definition is different from the definition used in the finite dimensional case. For more details, see [6].

Definition 2.3. *We say that the constraint qualification (MFCQ) is satisfied at* $x^* \in \Omega_0$ *if*

(i) There exist $\hat{\gamma} \in L_\infty([0,T];\mathbb{R}^n)$ *and* $\hat{b} > 0$ *such that, for almost every* $t \in [0,T]$,

$$\nabla h(x^*(t),t)^\top \hat{\gamma}(t) = 0 \quad \textit{and} \quad \nabla g_j(x^*(t),t)^\top \hat{\gamma}(t) \geq \beta, \ j \in I_{\hat{b}}(t),$$

for some $\beta > 0$*;*

(ii) There exists $K > 0$ *such that, for almost every* $t \in [0,T]$,

$$\det\{\nabla h(x^*(t),t)\nabla h(x^*(t),t)^\top\} \geq K.$$

For $x \in \mathbb{R}^n$, let

$$\begin{cases} f_0(x,t) := -\int_0^T \nabla\phi(x^*(t),t)^\top \bar{H}(t)x dt < 0, \\ f_j(x,t) := -g_j(x^*(t),t) - \delta_j^{\hat{b}}(t)\nabla g_j(x^*(t),t)^\top \bar{H}(t)\gamma \leq 0, \ j \in J, \end{cases} \tag{2.2}$$

where $\hat{b}$ is given in (H4) and

$$\bar{H}(t) = I_n - \nabla h(x^*(t),t)^\top [\nabla h(x^*(t),t)\nabla h(x^*(t),t)^\top]^{-1} \nabla h(x^*(t),t), \ \text{a.e. in } [0,T].$$

Above, I_n denotes the identity matrix of order n.

$$\begin{aligned} F_0(x,t) \quad &= \quad -\int_0^T \nabla\phi(x^*(t),t)^\top x \, dt < 0, \\ F_i(x,t) \quad &= \quad -g_i(x^*(t),t) - \delta_i^b(t)\nabla g_i(x^*(t),t)^\top x \leq 0, \ i \in I, \end{aligned} \tag{2.3}$$

be a system corresponding to Problem (2.1) and

$$\mathscr{I}(x,t) = \{j \mid F_j(x,t) = \max\{F_0(x,t), F_1(x,t), \ldots, F_m(x,t)\}\}, \ t \in [0,T], \ x \in \mathbb{R}^n.$$

Definition 2.4. *(Arutyunov et al. [2]) System (2.3) is said to be regular when there exist a function $\bar{x}(\cdot) \in L_\infty([0,T];\mathbb{R}^n)$, real numbers $R \geq 0$ and $\alpha > 0$ such that for a. e. $t \in [0,T]$ and for all $x \in \mathbb{R}^n$ with $\|x - \bar{x}(t)\| \geq R$, there exists a vector $e = e(x,t) \in \mathbb{R}^n$ with $\|e\| = 1$, satisfying*

$$\langle \partial_x F_j(x,t), e \rangle \geq \alpha, \;\; \forall\, j \in \mathscr{I}(x,t),$$

where $\partial_x F_j$ denotes the partial subdifferential of F_j at (x,t) with respect to x in the sense of convex analysis. For more information on convex analysis, the reader is referred to Rockafellar [17].

Proposition 2.2. *(Theorem 3.8 in do Monte and de Oliveira [6]) Assume that (H1)-(H4) hold and (MFCQ) is satisfied at $x^* \in \Omega_0$ which is a local optimal solution for (2.1). If the system (2.3) is regular, then there exists $(\tilde{u},\ \tilde{v}) \in L_\infty([0,T];\mathbb{R}_p \times \mathbb{R}^m_+)$ such that, for almost every $t \in [0,T]$, $\tilde{v}_i(t) \geq 0,\ i \in I$, and*

$$\nabla\phi(x^*(t),t) + \sum_{j=1}^{p} \tilde{u}_j(t)\nabla h_j(x^*(t),t) + \sum_{i=1}^{m} \tilde{v}_i(t)\nabla g_i(x^*(t),t) = 0.$$

3 VARIATIONAL-TYPE INEQUALITIES PROBLEM

The continuous-time complementarity problem $CTCP(f,K)$ is posed as to find x in $L_\infty([0,T];\mathbb{R}^n)$ such that, for a.e. $t \in [0,T]$, we have that

$$x(t) \in K, \qquad f(x(t),t) \in K^\circ \qquad \text{and} \qquad x(t)^\top f(x(t),t) = 0,$$

where $K \subset \mathbb{R}^n$ is a nonempty closed convex cone with vertex at 0, namely, if $x \in K$, $\alpha x \in K$ for all $\alpha > 0$. The polar cone K° of K is given by

$$K^\circ = \{y \in \mathbb{R}^n \mid y^\top x \geq 0,\ \forall\, x \in K\}.$$

$f : \mathbb{R}^n \times [0,T] \to \mathbb{R}^n$ is a nonlinear function with $f_j(\cdot,t)$, $j = 1,\ldots,n$, twice continuously differentiable throughout $[0,T]$ and $f_j(x,\cdot)$, $\nabla f_j(x,\cdot)$ and $\nabla^2 f_j(x,\cdot)$ measurable and essentially bounded for all $x \in \mathbb{R}^n$, $j = 1,\ldots,n$. Let

$$\Omega = \{x \in L_\infty([0,T];\mathbb{R}^n) \mid x(t) \in K \text{ a.e. } t \in [0,T]\}.$$

Definition 3.5. *The Variational-type Inequalities Problem $VIP(f,\Omega)$ consists in finding $x^* \in \Omega$ such that*

$$\int_0^T f(x^*(t),t)^\top (x(t) - x^*(t))\, dt \geq 0, \textit{ for all } x \in \Omega.$$

Lemma 3.1. *x^* is a solution of the $CTCP(f,K)$ if, and only if, x^* is a solution of $VIP(f,\Omega)$.*

Proof. If $x^* \in L_\infty([0,T],\mathbb{R}^n)$ is a solution of $CTCP(f,K)$ then $f(x^*(t),t) \in K^\circ$ a.e. $t \in [0,T]$, and we can conclude that

$$x(t)^\top f(x^*(t),t) \geq 0, \text{ a.e. } t \in [0,T], \text{ for all } x \in \Omega. \tag{3.1}$$

Using (3.1) and the hypothesis, for a.e. $t \in [0,T]$ and for all $x \in \Omega$, we have that

$$\begin{aligned} f(x^*(t),t)^\top(x(t)-x^*(t)) &= f(x^*(t),t)^\top x(t) - f(x^*(t),t)^\top x^*(t) \\ &= f(x^*(t),t)^\top x(t) \\ &\geq 0, \end{aligned}$$

resulting that $x^* \in \Omega$ is a solution of $VIP(f,\Omega)$. Conversely, if $x^* \in \Omega$ is a solution of $VIP(f,\Omega)$, then $x^*(t) \in K$ a.e. $t \in [0,T]$. The inequality in Definition 3.5 holds for all $x \in \Omega$. Particularly, for $x = 0 \in \Omega$ and $x = 2x^* \in \Omega$ we have that

$$\int_0^T f(x^*(t),t)^\top x^*(t)\, dt \leq 0 \qquad \text{and} \qquad \int_0^T f(x^*(t),t)^\top x^*(t)\, dt \geq 0,$$

respectively, resulting in

$$\int_0^T f(x^*(t),t)^\top x^*(t)\, dt = 0. \tag{3.2}$$

Statement: For all $x \in \Omega$, $f(x^*(t),t)^\top x(t) \geq 0$ a.e. $t \in [0,T]$. Indeed, suppose that there exists $\tilde{x} \in \Omega$ and a subset $D \subset [0,T]$, with positive measure, such that $f(x^*(t),t)^\top \tilde{x}(t) < 0$ for all $t \in D$. Define $\bar{x} \in \Omega$ given by

$$\bar{x}(t) = \begin{cases} \tilde{x}(t) \; if \; t \in D, \\ 0 \quad if \; t \in [0,T] \setminus D. \end{cases}$$

Then, using (3.2) and the definition of $\bar{x}$, we have that

$$\begin{aligned} & \int_0^T f(x^*(t),t)^\top (\bar{x}(t) - x^*(t))\, dt \\ = \; & \int_0^T f(x^*(t),t)^\top \bar{x}(t)\, dt - \int_0^T f(x^*(t),t)^\top x^*(t)\, dt \\ = \; & \int_D f(x^*(t),t)^\top \tilde{x}(t)\, dt \\ < \; & 0, \end{aligned}$$

contradicting the fact that x^* is a solution of $VIP(f,\Omega)$. Therefore, by the above statement, $f(x^*(t),t) \in K^\circ$ a.e. $t \in [0,T]$. Besides that, as $f(x^*(t),t)^\top x^*(t) \geq 0$ a.e. $t \in [0,T]$, it results from (3.2) that $f(x^*(t),t)^\top x^*(t) = 0$ a.e. $t \in [0,T]$, concluding the proof. □

Now, consider the auxiliary continuous-time problem

$$\begin{array}{ll} \text{maximize} & P(x) = -\displaystyle\int_0^T f(x(t),t)^\top x(t)\, dt \\ \text{subject to} & x(t) \in K \text{ a.e. } t \in [0,T], \\ & f(x(t),t) \in K^\circ \text{ a.e. } t \in [0,T], \\ & x \in L_\infty([0,T],\mathbb{R}^n). \end{array} \tag{3.3}$$

Proposition 3.3. *$x^* \in \Omega$ is a solution of Problem $CTCP(f,K)$ if, and only if, x^* is a global maximum point of Problem (3.3) with $P(x^*) = 0$.*

Proof. If x^* is a solution of $CTCP(f,K)$, then $x^*(t) \in K$, $f(x^*(t),t) \in K^\circ$ a.e. $t \in [0,T]$ and

$$P(x^*) = -\int_0^T f(x^*(t),t)^\top x^*(t)\, dt = 0.$$

Let $x \in L_\infty([0,T],\mathbb{R}^n)$ a feasible point of (3.3). Then, $x(t) \in K$ a.e. $t \in [0,T]$ and $f(x(t),t) \in K^\circ$ a.e. $t \in [0,T]$, imply that

$$P(x) = -\int_0^T f(x(t),t)^\top x(t)\, dt \leq 0.$$

Therefore, $P(x) \leq P(x^*)$ for all $x \in \Omega$ and x^* is a global maximum point of (3.3) with $P(x^*) = 0$. Conversely, if x^* is a global maximum point of (3.3), then $x^*(t) \in K$ a.e. $t \in [0,T]$, $f(x^*(t),t) \in K^\circ$ a.e. $t \in [0,T]$ and $P(x^*) = 0$. If $x \in \Omega$, from definition of polar cone K° we obtain

$$-\int_0^T f(x^*(t),t)^\top x(t)\, dt \leq 0 = P(x^*).$$

Therefore,

$$\begin{aligned} & -\int_0^T f(x^*(t),t)^\top x(t)\, dt \leq -\int_0^T f(x^*(t),t)^\top x^*(t)\, dt \\ \Leftrightarrow & \int_0^T f(x^*(t),t)^\top (x(t) - x^*(t))\, dt \geq 0, \end{aligned}$$

that is, x^* is a solution of $VIP(f,\Omega)$ and, by Lemma 3.1, x^* is a solution of $CTCP(f,K)$. □

Remark 1. *If x^* is a global maximum point of (3.3) and a regularity condition is verified on the constraint set Ω, then optimality conditions which are found in the literature can be applied to solve (3.3). In our case, we will use the Proposition 2.1 and Proposition 2.2. The next example illustrates the Proposition 3.3.*

Example 3.1. *Consider Problem $CTCP(f,K)$ with $x : [0,1] \to \mathbb{R}$, $f(x,t) = [x]^2 - tx$ and $K = \mathbb{R}_+ = K^\circ$. Then, (3.3) is given as*

$$\begin{aligned} \textit{maximize} \quad & P(x) = -\int_0^1 \left\{ [x(t)]^2 - tx(t) \right\} x(t)\, dt \\ \textit{subject to} \quad & x(t) \geq 0 \textit{ a.e. } t \in [0,1], \\ & [x(t)]^2 - tx(t) \geq 0 \textit{ a.e. } t \in [0,1], \\ & x \in L_\infty([0,T],\mathbb{R}^n). \end{aligned}$$

Note that the two constraints are satisfied when $x(t) - t \geq 0$ a.e. $t \in [0,T]$, and the problem can be written in the form

$$\begin{aligned} \textit{maximize} \quad & P(x) = -\int_0^1 \left\{ [x(t)]^2 - tx(t) \right\} x(t)\, dt \\ \textit{subject to} \quad & x(t) - t \geq 0 \textit{ a.e. } t \in [0,1], \\ & x \in L_\infty([0,1],\mathbb{R}^n). \end{aligned} \tag{3.4}$$

If x^* is a global maximum point of (3.4), then the Proposition 2.1 guarantees us that there exists $u^* \in L_\infty([0,1];\mathbb{R})$ such that, for a.e. $t \in [0,1]$,

(i) $-3[x^*(t)]^2 + 2tx^*(t) + u^*(t) = 0 \Rightarrow u^*(t) = 3[x^*(t)]^2 - 2tx^*(t)$,

(ii) $u^*(t) \geq 0$ *and* $u^*(t)[x^*(t) - t] = 0$,

resulting that $x^*(t) = t$ *and* $u^*(t) = t^2$ *for a.e.* $t \in [0,1]$*, with* $P(x^*) = 0$*. Observe that* $x^*(t) = t$ *a.e.* $t \in [0,T]$ *is a candidate solution to the problem (3.4), but it is a solution of* $CTCP(f,K)$*. Then, by Proposition 3.3, we can guarantee that* x^* *is a global minimizer of (3.4).*

4 THE CASE $K = \mathbb{R}^n_+$

Let us consider K to be the positive octant of $\mathbb{R}^n$. In this case, $K = K^\circ$. The Fischer-Burmeister function (see [10]) $\varphi : \mathbb{R}^2 \to \mathbb{R}$ is given by

$$\varphi(a,b) = \sqrt{a^2 + b^2} - a - b. \tag{4.1}$$

This function has the property that $\varphi(a,b) = 0 \Leftrightarrow a \geq 0,\ b \geq 0,\ ab = 0$. The partial derivates of φ will be denoted by $\dfrac{\partial \varphi}{\partial a}(a,b)$ and $\dfrac{\partial \varphi}{\partial b}(a,b)$. Suppose that Problem (3.3) satisfy the assumptions (H1)-(H4) and (LICQ) at the local optimal solution x^*. Then, the Proposition 2.1 guarantees that there exist $u^*,\ v^* \in L_\infty([0,T];\mathbb{R}^n)$, with $u^*(t) \geq 0$ and $v^*(t) \geq 0$ a.e. $t \in [0,T]$, such that

$$\|f(x^*(t),t) + \sum_{i=1}^{n} \nabla f_i(x^*(t),t)[x_i^*(t) - v_i^*(t)] - u^*(t)\| = 0,$$

$$\varphi(x_i^*(t), u_i^*(t)) = 0 \quad \text{and} \quad \varphi(f_i(x_i^*(t),t), v_i^*(t)) = 0$$

for a.e. $t \in [0,T]$ and $i = 1,\ldots,n$. Simplifying the notation, define

$$\tilde{\varphi}(x^*(t),u^*(t)) = 2\begin{pmatrix} \varphi(x_1^*(t), u_1^*(t)) \\ \vdots \\ \varphi(x_n^*(t), u_n^*(t)) \end{pmatrix},$$

$$\tilde{\varphi}(f(x^*(t),t),v^*(t)) = 2\begin{pmatrix} \varphi(f_1(x^*(t),t), v_1^*(t)) \\ \vdots \\ \varphi(f_n(x^*(t),t), v_n^*(t)) \end{pmatrix},$$

$$\Phi_x(t) = \operatorname{diag}\left(\frac{\partial \varphi}{\partial a}(x_i^*(t), u_i^*(t))\right)_{i=1}^{n}, \quad \Phi_u(t) = \operatorname{diag}\left(\frac{\partial \varphi}{\partial b}(x_i^*(t), u_i^*(t))\right)_{i=1}^{n},$$

$$\Phi_v(t) = \operatorname{diag}\left(\frac{\partial \varphi}{\partial b}(f_i(x^*(t),t), v_i^*(t))\right)_{i=1}^{n}$$

and the $n \times n$ matrix $\Phi_f(t) = (c_{ij}(t))$ where

$$c_{ij}(t) = \frac{\partial \varphi}{\partial a}(f_j(x^*(t),t), v_j^*(t))\frac{\partial f_j}{\partial x_i}(x^*(t),t),\ i,j = 1,2,\ldots,n.$$

Also, we write

$$\nabla f(x(t),t)^\top = \begin{pmatrix} \nabla f_1(x(t),t) & \nabla f_2(x(t),t) & \dots & \nabla f_n(x(t),t) \end{pmatrix} \text{ a.e. } t \in [0,T].$$

Consider the following unconstrained continuous-time problem:

$$\begin{aligned} &\text{maximize} \quad Q(x,u,v) = -\int_0^T F(x(t),u(t),v(t),t)\,dt \\ &\text{subject to} \quad x,\ u,\ v \in L_\infty([0,T],\mathbb{R}^n), \end{aligned} \tag{4.2}$$

where $F : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times [0,T] \to \mathbb{R}$ is defined by

$$F(x,u,v,t) = \|f(x,t) + \nabla f(x,t)^\top [x - v] - u\|^2 + \sigma(x,u,v),$$

with $\sigma(x,u,v) = \sum_{i=1}^{n} [\varphi(x_i,u_i)]^2 + [\varphi(f_i(x,t),v_i)]^2$ a.e. $t \in [0,T]$.

Theorem 4.1. *If $(x^*,u^*,v^*) \in L_\infty([0,T],\mathbb{R}^n \times \mathbb{R}^n_+ \times \mathbb{R}^n_+)$ is a global optimal solution for Problem (4.2) with $Q(x^*,u^*,v^*) = 0$ and $[x^*(t) - v^*(t)] \in \mathrm{Ker}(\nabla f(x(t),t)^\top)$ a.e. $t \in [0,T]$, then x^* is a solution of Problem $CTCP(f,K)$.*

Proof. By definition of F, if $Q(x^*,u^*,v^*) = 0$ then $F(x^*(t),u^*(t),v^*(t),t) = 0$ a.e. $t \in [0,T]$. Then, $\sigma(x^*(t),u^*(t),v^*(t)) = 0$ a.e. $t \in [0,T]$ implies that

$$\begin{aligned} &\sum_{i=1}^{n} [\varphi(x_i^*(t),u_i^*(t))]^2 = 0 \text{ a.e. } t \in [0,T] \\ \Leftrightarrow \quad &\varphi(x_i^*(t),u_i^*(t)) = 0 \text{ a.e. } t \in [0,T],\ i = 1,\dots,n \\ \Leftrightarrow \quad &x_i^*(t) \geq 0,\ u_i^*(t) \geq 0,\ x_i^*(t)u_i^*(t) = 0 \text{ a.e. } t \in [0,T],\ i = 1,\dots,n. \end{aligned}$$

Similarly, we conclude that $f_i(x^*(t),t) \geq 0,\ v_i^*(t) \geq 0,\ f_i(x^*(t),t)v_i^*(t) = 0$ a.e. $t \in [0,T],\ i = 1,\dots,n$. Thus, we have that $x^*(t) \in K$ and $f(x^*(t),t) \in K^\circ$ a.e. $t \in [0,T]$. Moreover, since

$$f(x^*(t),t) + \nabla f(x^*(t),t)^\top [x^*(t) - v^*(t)] - u^*(t) = 0 \text{ a.e. } t \in [0,T],$$

and using the fact that $[x^*(t) - v^*(t)] \in \mathrm{Ker}(\nabla f(x^*(t),t)^\top)$ a.e. $t \in [0,T]$, it results that

$$x^*(t)^\top f(x^*(t),t) = -x^*(t)^\top \nabla f(x^*(t),t)^\top [x^*(t) - v^*(t)] + x^*(t)^\top u^*(t) = 0.$$

Therefore, x^* is a solution for Problem $CTCP(f,K)$. □

Theorem 4.2. *Assume that the Problem (3.3) satisfy the assumptions (H1)-(H4) and (LICQ) at the local optimal solution x^*. If $x^* \in L_\infty([0,T];\mathbb{R}^n)$ is a solution of Problem $CTCP(f,K)$, then there exist $u^*,\ v^*$ in $L_\infty([0,T];\mathbb{R}^n_+)$ such that (x^*,u^*,v^*) is a global optimal solution for Problem (4.2) with $Q(x^*,u^*,v^*) = 0$.*

Proof. If x^* is a solution of Problem $CTCP(f,K)$, by Proposition 3.3, x^* is a global maximum point of Problem (3.3). Using the Proposition 2.1, we conclude that $F(x^*(t),u^*(t),v^*(t),t) = 0$ a.e. $t \in [0,T]$, in other words, (x^*,u^*,v^*) is a global maximum point of (4.2) with $Q(x^*,u^*,v^*) = 0$. □

Most of the algorithms used in the resolution of Problem (4.2) guarantee convergence only to stationary points. A global minimum point is very hard to find (see [1]). For this purpose, we will derive conditions for stationary points of (4.2). We say that $p = (x,u,v) \in L_\infty([0,T],\mathbb{R}^{3n})$ is a stationary point of (4.2) if, and only if, $\nabla F(p(t),t) = 0$ a.e. $t \in [0,T]$.

Definition 4.6. *The $n \times n$ matrix $M(x,t)$, with elements $m_{ij}(x,t)$, $x \in \mathbb{R}^n$, $t \in [0,T]$, $i, j = 1,\ldots,n$, is positive definite at $x^* \in L_\infty([0,T];\mathbb{R}^n)$ if, for all $y \in L_\infty([0,T];\mathbb{R}^n)$ and a.e. $t \in [0,T]$,*

$$y(t)^\top M(x^*(t),t)y(t) > 0 \text{ whenever } y(t) \neq 0.$$

The next theorem relates stationary points of (4.2) to solutions of Problem $CTCP(f,K)$.

Theorem 4.3. *Let (x^*,u^*,v^*) is a stationary point of (4.2). Regarding the Problem (4.2), assume that*

(i) *$D(t)$ is definite positive a.e. $t \in [0,T]$, where*

$$D(t) = 2\nabla f(x^*(t),t) + \sum_{i=1}^{n} [x_i^*(t) - v_i^*(t)]\nabla^2 f_i(x^*(t),t) \text{ a.e. } t \in [0,T];$$

(ii) $[x^*(t) - v^*(t)] \in \mathrm{Ker}(\nabla f(x^*(t),t)^\top)$*;*

(iv) $[\Phi_u(t)\tilde{\varphi}(x^*(t),u^*(t))] \in K$ *and* $[\Phi_f(t)\tilde{\varphi}(f(x^*(t),t),v^*(t))] \in K^\circ$ *a.e.* $t \in [0,T]$.

Then x^* is a solution of $CTCP(f,K)$.

Proof. Note that, if $w(t) = f(x(t),t) + \nabla f(x(t),t)^\top[x(t) - v(t)] - u(t)$ a.e. $t \in [0,T]$, we have that

$$\begin{aligned}
&\nabla_x \left\{ \|f(x(t),t) + \nabla f(x(t),t)^\top [x(t) - v(t)] - u(t)\|^2 \right\} \\
&= 2\left\{ \nabla f(x(t),t) + \sum_{i=1}^{n} [x_i(t) - v_i(t)]\nabla^2 f_i(x(t),t) + \nabla f(x(t),t) \right\}^\top w(t) \\
&= 2\left\{ 2\nabla f(x(t),t) + \sum_{i=1}^{n} [x_i(t) - v_i(t)]\nabla^2 f_i(x(t),t) \right\}^\top w(t).
\end{aligned}$$

For a.e. $t \in [0,T]$, let us denote

$$w^*(t) = f(x^*(t),t) + \nabla f(x^*(t),t)^\top [x^*(t) - v^*(t)] - u^*(t).$$

If (x^*,u^*,v^*) is a stationary point of (4.2), then $\nabla F(x^*(t),u^*(t),v^*(t),t) = 0$ a.e. $t \in [0,T]$, that is, for almost every $t \in [0,T]$,

$$D(t)w^*(t) + \Phi_x(t)\tilde{\varphi}(x^*(t),u^*(t)) + \Phi_f(t)\tilde{\varphi}(f(x^*(t),t),v^*(t)) = 0, \tag{4.3}$$

$$-w^*(t) + \Phi_u(t)\tilde{\varphi}(x^*(t),u^*(t)) = 0, \tag{4.4}$$

$$-\nabla f(x^*(t),t)w^*(t) + \Phi_v(t)\tilde{\varphi}(f(x^*(t),t),v^*(t)) = 0. \tag{4.5}$$

From (4.3) and (4.4) we obtain, for almost every $t \in [0,T]$,

$$\begin{aligned} w^*(t)^\top D(t) w^*(t) \quad &+ \quad w^*(t)^\top \Phi_x(t) \tilde{\varphi}(x^*(t), u^*(t)) \\ &+ \quad w^*(t)^\top \Phi_f(t) \tilde{\varphi}(f(x^*(t),t), v^*(t)) = 0 \end{aligned} \tag{4.6}$$

and

$$w^*(t)^\top = \tilde{\varphi}(x^*(t), u^*(t))^\top \Phi_u(t), \tag{4.7}$$

respectively. Using (4.6) and (4.7) we have, for almost every $t \in [0,T]$, that

$$\begin{aligned} & w^*(t)^\top D(t) w^*(t) \\ = \quad & -\tilde{\varphi}(x^*(t), u^*(t))^\top \{\Phi_u(t)\Phi_x(t)\}\, \tilde{\varphi}(x^*(t), u^*(t)) \\ & -\tilde{\varphi}(x^*(t), u^*(t))^\top \left\{\Phi_u(t)\Phi_f(t)\right\} \tilde{\varphi}(f(x^*(t),t), v^*(t)). \end{aligned} \tag{4.8}$$

Noting that

$$\frac{\partial \varphi}{\partial a}(x_i^*(t), u_i^*(t)) \frac{\partial \varphi}{\partial b}(x_i^*(t), u_i^*(t)) \geq 0 \text{ a.e. } t \in [0,T],\ i = 1, \ldots, n,$$

we have that $\Phi_u(t)\Phi_x(t)$ is positive semi-definite for a.e. $t \in [0,T]$ and using assumption (iii) we conclude that $w^*(t)^\top D(t) w^*(t) \leq 0$ a.e. $t \in [0,T]$. But, from (i), it results that $w(t)^\top D(t) w(t) > 0$ for all $w \in L_\infty([0,T];\mathbb{R}^n)$, whenever $w(t) \not\equiv 0$. Then

$$w^*(t) = 0 \text{ a.e. } t \in [0,T]. \tag{4.9}$$

From (4.9), (ii) and the definition of $w^*(t)$, it follows that $u^*(t) = f(x^*(t),t)$ a.e. $t \in [0,T]$. Replacing (4.9) in (4.5), we obtain

$$\Phi_v(t) \tilde{\varphi}(f(x^*(t),t), v^*(t)) = 0 \Leftrightarrow \varphi(f_i(x^*(t),t), v_i^*(t)) \frac{\partial \varphi}{\partial b}(f_i(x^*(t),t), v_i^*(t)) = 0,$$

for $i = 1, \ldots, n$. So, for almost every $t \in [0,T]$ and for each $i = 1, \ldots, n$, we have that $\varphi(f_i(x^*(t),t), v_i^*(t)) = 0$ or $\dfrac{\partial \varphi}{\partial b}(f_i(x^*(t),t), v_i^*(t)) = 0$. Observe that

$$\frac{\partial \varphi}{\partial b}(f_i(x^*(t),t), v_i^*(t)) = 0 \Rightarrow \frac{v_i^*(t)}{\sqrt{[f_i(x^*(t),t)]^2 + [v_i^*(t)]^2}} - 1 = 0$$

implying that $f_i(x^*(t),t) = 0$ and $v_i^*(t) > 0$, that is, $\varphi(f_i(x^*(t),t), v_i^*(t)) = 0$. Therefore,

$$\tilde{\varphi}(f(x^*(t),t), v^*(t)) = 0 \text{ a.e. } t \in [0,T]. \tag{4.10}$$

It results from (4.10) that $f(x^*(t),t) \geq 0$ a.e. $t \in [0,T]$, that is, $f(x^*(t),t) \in K^\circ$ a.e. $t \in [0,T]$. With a similar argument, replacing (4.9) in (4.4), we obtain $x^*(t) \geq 0$, $u^*(t) \geq 0$, and $x^*(t)^\top u^*(t) = 0$ a. e. $t \in [0,T]$, namely $x^*(t) \in K$ a.e. $t \in [0,T]$. Remembering that $u^*(t) = f(x^*(t),t)$ a.e. $t \in [0,T]$, we conclude that $x^*(t)^\top f(x^*(t),t) = 0$ for a.e. $t \in [0,T]$. □

Example 4.1. *Considering Example 3.1, note that the* $(t,0,t)$ *a.e.* $t \in [0,1]$ *is a stationary point for*

$$\begin{array}{ll} \textit{maximize} & Q(x,u,v) = -\displaystyle\int_0^1 F(x(t),u(t),v(t),t)\ dt \\ \textit{subject to} & x,\ u,\ v \in L_\infty([0,T],\mathbb{R}), \end{array}$$

where

$$F(x(t),u(t),v(t),t) = \left|[x(t)]^2 - tx(t) + (2x(t)-t)[x(t)-v(t)] - u(t)\right|^2 + \sigma(x(t),u(t),v(t)),$$

satisfying all assumptions of Theorem 4.3. Then, $x^*(t) = t$ *a.e.* $t \in [0,1]$ *is a solution of* $CTCP(f,K)$.

5 THE SECOND-ORDER CONE CASE

In this section, we will consider the second-order cone defined as

$$K = \left\{ x \in \mathbb{R}^n \mid x_1 \geq \left(\sum_{i=2}^{n} a_i^2 x_i^2 \right)^{\frac{1}{2}} \right\}, \tag{5.1}$$

where $a_i \in \mathbb{R},\ i = 2,\dots,n$, and its polar cone given by

$$K^\circ = \left\{ x \in \mathbb{R}^n \mid x_1 \geq \left(\sum_{\substack{i=2 \\ i\notin J}}^{n} \frac{1}{a_i^2} x_i^2 \right)^{\frac{1}{2}},\ x_i = 0 \text{ for } i \in J \right\}, \tag{5.2}$$

where $J = \{ j \in \{2,\dots,n\} \mid a_j = 0\}$. Our goal is to write K.K.T. optimality conditions for the problem (3.3) with

$$\Omega = \{ x \in L_\infty([0,T];\mathbb{R}^n) \mid x(t) \in K \text{ a.e. } t \in [0,T] \}$$

and K is the second-order cone. But, note that the constraint of this problem is non-differentiable. To work around this problem, let us rewrite (5.1) as

$$K = \left\{ x \in \mathbb{R}^n \mid x_1^2 \geq \sum_{i=2}^{n} a_i^2 x_i^2,\ x_1 \geq 0 \right\} \tag{5.3}$$

and (5.2) as

$$K^\circ = \left\{ x \in \mathbb{R}^n \mid x_1^2 \geq \sum_{\substack{i=2 \\ i\notin J}}^{n} \frac{1}{a_i^2} x_i^2,\ x_1 \geq 0,\ x_i = 0 \text{ for } i \in J \right\}. \tag{5.4}$$

Remark 1. *In definition of* K°*, the condition "*$x_i = 0$ *for* $i \in J$*" is essential. For example, consider*

$$K = \{ x \in \mathbb{R}^3 \mid x_1 \geq 4x_2^2, x_1 \geq 0\} \quad \textit{and} \quad K^\circ = \{ x \in \mathbb{R}^3 \mid x_1^2 \geq \frac{1}{4} x_2^2,\ x_1 \geq 0\}.$$

Note that $x = (0,0,1) \in K$, $y = (0,0,-1) \in K^\circ$, but $x^\top y = -1$ contradicting the definition of polar cone. With the purpose of applying Mangasarian-Fromovitz constraint qualification, we will assume that $a_i \neq 0$, $i = 1,2,\dots,n$, in the definition of K.

Define the diagonal $n \times n$ matrices

$$A = \operatorname{diag}(1, -a_2^2, \dots, -a_n^2) \quad \text{and} \quad \bar{A} = \operatorname{diag}\left(1, -\frac{1}{a_2^2}, \dots, -\frac{1}{a_n^2}\right).$$

In the matrix form, with $e_1 = (1,0,\dots,0)^\top \in \mathbb{R}^n$, (5.3) and (5.4) become

$$K = \left\{ x \in \mathbb{R}^n \mid \frac{1}{2} x^\top A x \geq 0,\ e_1^\top x \geq 0 \right\} \tag{5.5}$$

and

$$K^\circ = \left\{ x \in \mathbb{R}^n \mid \frac{1}{2} x^\top \bar{A} x \geq 0,\ e_1^\top x \geq 0 \right\}. \tag{5.6}$$

Then, for fixed $x^* \in \Omega$, Problem (3.3) is given by

$$\begin{array}{ll} \text{maximize} & P(x) = -\displaystyle\int_0^T f(x(t),t)^\top x(t)\, dt \\ \text{subject to} & \frac{1}{2} x(t)^\top A x(t) \geq 0 \text{ a.e. in } [0,T], \\ & e_1^\top x(t) \geq 0 \text{ a.e. in } [0,T], \\ & \frac{1}{2} f(x(t),t)^\top \bar{A} f(x(t),t) \geq 0 \text{ a.e. in } [0,T], \\ & e_1^\top f(x(t),t) \geq 0 \text{ a.e. in } [0,T]. \end{array} \tag{5.7}$$

Note that Problem (5.7) is a particular case of (2.1), with

$$\phi(x,t) = -f(x,t)^\top x, \quad g_1(x,t) = \frac{1}{2} x^\top A x, \quad g_2(x,t) = e_1^\top x,$$

$$g_3(x,t) = \frac{1}{2} f(x,t)^\top \bar{A} f(x,t), \quad g_4(x,t) = e_1^\top f(x,t),$$

and satisfy (H1)-(H4). To ensure regularity of the constraints, we will assume that Problem (5.7) satisfies MFCQ.

Now, we will define a unconstrained problem related to $CTCP(f,K)$, where K is the second-order cone. To this end, we will use the Fischer-Burmeister function (4.1) again. For almost every $t \in [0,T]$, define

$$g(x(t),t) = \begin{pmatrix} g_1(x(t),t) \\ g_2(x(t),t) \\ g_3(x(t),t) \\ g_4(x(t),t) \end{pmatrix}, \quad \nabla g(x(t),t) = \begin{pmatrix} \nabla g_1(x(t),t)^\top \\ \nabla g_2(x(t),t)^\top \\ \nabla g_3(x(t),t)^\top \\ \nabla g_4(x(t),t)^\top \end{pmatrix},$$

and consider $u,\ z$ functions in $L_\infty([0,T];\mathbb{R}^4)$ with

$$u(t) = \begin{pmatrix} u_1(t) \\ u_2(t) \\ u_3(t) \\ u_4(t) \end{pmatrix} \quad \text{and} \quad z(t) = \begin{pmatrix} z_1(t) \\ z_2(t) \\ z_3(t) \\ z_4(t) \end{pmatrix},$$

and $v \in L_\infty([0,T],\mathbb{R}^n)$. Define the following unconstrained continuous-time problem:

$$\begin{array}{ll} \text{maximize} & R(x,u,z) = -\int_0^T G(x(t),u(t),z(t),t)\ dt, \\ \text{subject to} & u,\ z \in L_\infty([0,T],\mathbb{R}^4), \\ & x \in L_\infty([0,T],\mathbb{R}^n), \end{array} \tag{5.8}$$

where

$$G(x,u,z,t) = \|f(x,t) + \nabla f(x,t)^\top x - \nabla g(x,t)^\top u\|^2 + \|g(x,t) - z\|^2 + \sigma(u,z)$$

and

$$\sigma(u(t),z(t)) = \sum_{i=1}^{4} [\varphi_i(u_i(t),z_i(t))]^2 \text{ a.e. } t \in [0,T],$$

with φ_i representing Fischer-Burmeister functions at $(u_i(t),z_i(t))$ a.e. $t \in [0,T]$ and $i \in \{1,2,3,4\}$.

Theorem 5.4. *If $(x^*,u^*,z^*) \in L_\infty([0,T];\mathbb{R}^n \times \mathbb{R}^4 \times \mathbb{R}^4)$ is a global optimal solution of (5.8) with $R(x^*,u^*,z^*) = 0$ and $[x^*(t) - \bar{A}f(x^*(t),t)u_3^*(t) - e_1 u_4^*(t)] \in \mathrm{Ker}(\nabla f(x^*(t),t)^\top)$, then x^* is a solution of $CTCP(f,K)$.*

Proof. By hypothesis, for almost every $t \in [0,T]$, we have that

(a) $f(x^*(t),t) + \nabla f(x^*(t),t)^\top x^*(t) - \nabla g(x^*(t),t)^\top u^*(t) = 0;$

(b) $\frac{1}{2}x^*(t)^\top A x^*(t) - z_1^*(t) = 0;$

(c) $e_1^\top x^*(t) - z_2^*(t) = 0;$

(d) $\frac{1}{2}f(x^*(t),t)^\top \bar{A} f(x^*(t),t) - z_3^*(t) = 0;$

(e) $e_1^\top f(x^*(t),t) - z_4^*(t) = 0;$

(f) $\sigma(u^*(t),z^*(t)) = 0.$

Using the property of the Fischer-Burmeister function, for almost every $t \in [0,T]$, from (f) it results

$$\varphi_i(u_i^*(t),z_i^*(t)) = 0 \quad \Rightarrow \quad u_i^*(t) \geq 0,\ z_i^*(t) \geq 0 \text{ and } u_i^*(t)z_i^*(t) = 0,\ i = 1,2,3,4.$$

As $z_1^*(t) \geq 0$ and $z_2^*(t) \geq 0$ a.e. $t \in [0,T]$, from (b) and (c) it results that

$$\frac{1}{2} x^*(t)^\top A x^*(t) = z_1^*(t) \geq 0 \text{ a.e. in } [0,T], \tag{5.9}$$

$$x_1^*(t) = z_2^*(t) \geq 0 \text{ a.e. in } [0,T], \tag{5.10}$$

so that $x^*(t) \in K$ a.e. $t \in [0,T]$. Analogously, as $z_3^*(t) \geq 0$ and $z_4^*(t) \geq 0$ a.e. $t \in [0,T]$, from (d) and (e) it results that $f(x^*(t),t) \in K^\circ$ a.e. $t \in [0,T]$. Now, for each $t \in [0,T]$,

(i) if $x_1^*(t) = 0$, it results from (5.9) that $x_i^*(t) = 0$, $2 \leq i \leq n$. Therefore, we have that $x^*(t)^\top f(x^*(t),t) = 0$.

(ii) if $x_1^*(t) > 0$, from (c) we have that $z_2^*(t) > 0$; from (f), we see that this implies in $u_2^*(t) = 0$. Therefore, using (a), we conclude that

$$\begin{aligned} f(x^*(t),t) = \quad & -\nabla f(x^*(t),t)^\top x^*(t) + A x^*(t) u_1^*(t) \\ & + \nabla f(x^*(t),t)^\top \bar{A} f(x^*(t),t) u_3^*(t) + \nabla f_1(x^*(t),t) u_4^*(t) \end{aligned}$$

or

$$\begin{aligned} f(x^*(t),t) = \quad & -\nabla f(x^*(t),t)^\top \left[x^*(t) - \bar{A} f(x^*(t),t) u_3^*(t) - e_1 u_4^*(t)\right] \\ & + A x^*(t) u_1^*(t). \end{aligned}$$

Multiplying the left side by $x^*(t)^\top$, it results that

$$\begin{aligned} & x^*(t)^\top f(x^*(t),t) \\ = \quad & -x^*(t)^\top \nabla f(x^*(t),t)^\top \left[x^*(t) - \bar{A} f(x^*(t),t) u_3^*(t) - e_1 u_4^*(t)\right] \\ & + x^*(t)^\top A x^*(t) u_1^*(t) \\ = \quad & 0 \end{aligned}$$

because $x^*(t)^\top A x^*(t) u_1^*(t) = 2 z_1^*(t) u_1^* = 0$ and $[x^*(t) - \bar{A} f(x^*(t),t) u_3^*(t) - e_1 u_4^*(t)] \in \operatorname{Ker}(\nabla f(x^*(t),t)^\top)$ a.e. $t \in [0,T]$.

From (i) and (ii) we can conclude that $x^*(t)^\top f(x^*(t),t) = 0$ for almost every $t \in [0,T]$. □

In the next example, we illustrate the use of Theorem 5.4.

Example 5.1. *We want to solve $CTCP(f,K)$ with*

$$f(x,t) = (x_3^2 + t, -x_2, x_1 - x_2) \textit{ and}$$

$$K = K^\circ = \{x \in \mathbb{R}^3 \mid x_1^2 \geq x_2^2 + x_3^2,\ x_1 \geq 0\}.$$

If $(x^*,u^*,z^*) \in L_\infty([0,T];\mathbb{R}^3\times\mathbb{R}^4\times\mathbb{R}^4)$ *is a solution of (5.8) with* $R(x^*,u^*,z^*)=0$, *where*

$$\begin{aligned} g_1(x,t) &= \frac{1}{2}(x_1^2-x_2^2-x_3^2),\\ g_2(x,t) &= x_1,\\ g_3(x,t) &= \frac{1}{2}[(x_3^2+t)^2-x_2^2-(x_1-x_2)^2],\\ g_4(x,t) &= x_3^2+t, \end{aligned}$$

then for almost every $t\in[0,T]$*, we have that*

(a) $x_3^*(t)^2+t+x_3^*(t)-x_1^*(t)u_1^*(t)-u_2^*(t)+(x_1^*(t)-x_2^*(t))u_3^*(t)=0$*;*

(b) $-x_2^*(t)-x_2^*(t)-x_3^*(t)+x_2^*(t)u_1^*(t)-(x_1^*(t)-2x_2^*(t))u_3^*(t)=0$*;*

(c) $x_1^*(t)-x_2^*(t)+2x_1^*(t)x_3^*(t)+x_3^*(t)u_1^*(t)-2(x_3^*(t)^2+t)x_3^*(t)u_3^*(t)-2x_3^*(t)u_4^*(t)=0$*;*

(d) $u_i^*(t)\geq 0,\ g_i(x^*(t),t)\geq 0,\ u_i^*(t)g_i(x^*(t),t)=0,\ i\in\{1,2,3,4\}$.

Observe that (x^*,u^*,z^*) *satisfies (a)-(d) with* $x^*(t)=(t,t,0)$*,* $u^*(t)=(1,0,1,0)$ *and* $z^*(t)=g(x^*(t),t)$ *a.e.* $t\in[0,T]$ *and is a global optimal solution of (5.8). Moreover,*

$$[x^*(t)-\bar{A}f(x^*(t),t)u_3^*(t)-e_1u_4^*(t)]\in \mathrm{Ker}(\nabla f(x^*(t),t)^\top)$$

is satisfied for a.e. $t\in[0,T]$*. Then, by Theorem 5.4,* x^* *is a solution for CTCP*(f,K)*, that is,* $x^*(t)\in K$*,* $f(x^*(t),t)\in K^\circ$ *and* $x^*(t)^\top f(x^*(t),t)=0$ *a.e.* $t\in[0,1]$.

Theorem 5.5. *Let* x^* *be solution of CTCP*(f,K) *and suppose that the data of Problem (5.7) satisfies (H1)-(H4) and Definition 2.3 hold and that the system (2.3) is regular. Then there exist* u^* *and* z^* *in* $L_\infty([0,T];\mathbb{R}^4_+)$ *such that* (x^*,u^*,z^*) *is a global optimal solution of (5.8), with* $R(x^*,u^*,z^*)=0$.

Proof. The assumptions guarantee that Proposition 2.2 can be applied to Problem (5.7), that is, that there exists u^* in $L_\infty([0,T];\mathbb{R}^4)$ such that, for almost every $t\in[0,T]$, we have

(a) $f(x^*(t),t)+\nabla f(x^*(t),t)^\top x^*(t)-\nabla g(x^*(t),t)^\top u^*(t)=0$;

(b) $g_1(x^*(t),t)=\frac{1}{2}x^*(t)^\top Ax^*(t)\geq 0$;

(c) $g_2(x^*(t),t)=e_1^\top x^*(t)\geq 0$;

(d) $g_3(x^*(t),t)=\frac{1}{2}f(x^*(t),t)^\top\bar{A}f(x^*(t),t)\geq 0$;

(e) $g_4(x^*(t),t)=e_1^\top f(x^*(t),t)\geq 0$;

(f) $u_i^*(t)\geq 0,\ g_i(x^*(t),t)\geq 0,\ u_i^*(t)g_i(x^*(t),t)=0,\ i\in\{1,2,3,4\}$;

From (a) it results

$$\|f(x^*(t),t)+\nabla f(x^*(t),t)x^*(t)-\nabla g(x^*(t),t)u^*(t)\|=0$$

a.e. in $[0,T]$. Defining $z_i^*(t)=g_i(x^*(t),t)$, $i=1,2,3,4$, it follows that $\|g(x^*(t),t)-z^*(t)\|=0$ a.e. in $[0,T]$. From (f) we obtain $\varphi_i(u_i^*(t),z_i^*(t))=0$ a.e. in $[0,T]$. Therefore, we can conclude that (x^*,u^*,z^*) is a global optimal solution of (5.8), with $R(x^*,u^*,z^*)=0$. $\square$

6 FINAL COMMENTS

In this work, we presented an approach to the resolution of the continuous-time complementarity problem by reformulating it as an equivalent unconstrained optimization problem. The definition and properties of the VIP(f,Ω) given by Zalmai in [22] are applied on the resolution of the continuous-time nonlinear complementarity problem presented by Bodo an Hanson in [3].

We proved that a solution of Problem $CTCP(f,K)$ is a global minimizer of Problem (3.3) with zero objective function value and vice versa. A discretization approach can be used to solve linear Problem (3.3) (for example, see [16, 18, 19, 20]).

For the polyhedral cone $K=\mathbb{R}^n_+$, we showed that a stationary point of the unconstrained Problem (4.2) with zero objective function value is a solution of Problem $CTCP(f,K)$. Moreover, we verified that stationary points of Problem (4.2) also are solutions of $CTCP(f,K)$.

The Fischer-Burmeister function was used to write an unconstrained auxiliary problem and to obtain solutions of $CTCP(f,K)$ when K is the second order cone. In this case, to ensure that some constraint qualification holds, assumptions over the constraints should be made (see Remark 1).

This article opens new perspectives for research in the area, such as the possibility of studying the generalized continuous-time nonlinear complementarity problem that consists of find $x\in L_\infty([0,T],\mathbb{R}^n)$ such that

$$F(x,t)\in K,\quad G(x,t)\in K^\circ \quad \text{and} \quad F(x,t)^\top G(x,t)=0,$$

where $F:\mathbb{R}^n\times[0,T]\to\mathbb{R}^n$, $G:\mathbb{R}^n\times[0,T]\to\mathbb{R}^n$, $K\subset\mathbb{R}^n$ is a nonempty closed convex cone with vertex at 0 and K° is the polar cone of K.

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