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Asymptotic Behaviour of a Viscoelastic Transmission Problem with a Tip Load

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ABSTRACT. We consider a transmission problem for a string composed by two components: one of them is a viscoelastic material (with viscoelasticity of memory type), and the other is an elastic material (without dissipation effective over this component). Additionally, we consider that in one end is attached a tip load. The main result is that the model is exponentially stable if and only if the memory effect is effective over the string. When there is no memory effect, then there is a lack of exponential stability, but the tip load produces a polynomial rate of decay. That is, the tip load is not strong enough to stabilize exponentially the system, but produces a polynomial rate of decay.

Keywords: transmission problems, memory effect, lack of exponential stability, tip load, hybrid system.

1 INTRODUCTION

We consider the transmission problem for the damped vibrations of a string, whose left end is rigidly attached and in the other end has an attached hollow-tip body that contains granular material. The string is composed by two components: one of them is a viscoelastic material (with viscoelasticity of memory type) and the other is an elastic material (without dissipation effective over this component).

More precisely, let us denote by U the displacement of the string. That is

$$U(x) = \begin{cases} u(x), & x \in \,]0, l_0[\\ v(x), & x \in \,]l_0, l[\end{cases}$$

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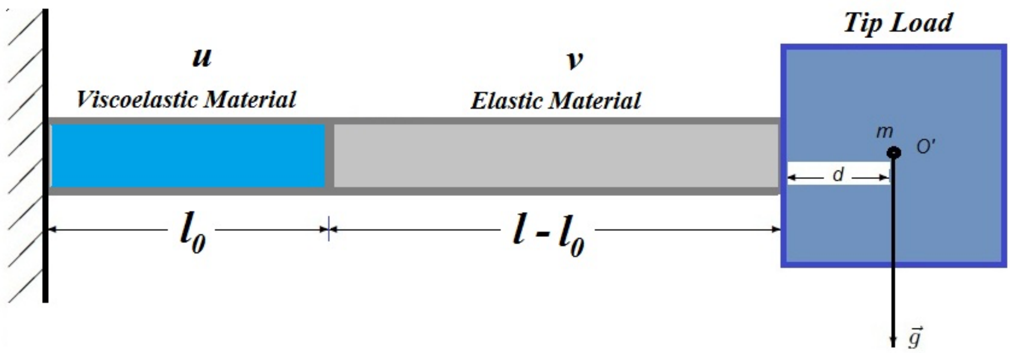

Figure 1: String with Tip Load.

where l is the length of the string and l_0 is the transmission point. The model that we consider in this paper is written as follows.

$$\rho_1 u_{tt} - \alpha_1 u_{xx} + \int_0^t g(t-s) u_{xx}(\cdot,s) ds = 0 \qquad \text{in} \quad]0,l_0[\times]0,+\infty[\tag{1.1}$$

$$\rho_2 v_{tt} - \alpha_2 v_{xx} = 0 \qquad \text{in} \quad]l_0,l[\times]0,+\infty[. \tag{1.2}$$

Here, $g : [0,+\infty) \to \mathbb{R}$ be the relaxation function, and α_1, α_2, ρ_1, ρ_2 are positive constants that reflect physical properties of the string. The boundary conditions are given by

$$u(0,t) = 0, \qquad v(l,t) = w(t), \qquad \forall \ t \geq 0, \tag{1.3}$$

and the transmission conditions are given by

$$u(l_0,t) = v(l_0,t), \quad \alpha_1 u_x(l_0,t) - \int_0^t g(t-s) u_x(l_0,s) ds = \alpha_2 v_x(l_0,t), \quad \forall t \geq 0. \tag{1.4}$$

We turn to model the motion of the right end with the attached tip body. We assume that the container is rigidly attached to the end $x = l$, and that the container and its contents have mass m and a center of mass O' located at distance d from the end of the string. We assume that the damping effect of the internal granular material can be represented by damping coefficient γ_1, whose precise contributions are described below

$$\rho_3 w_{tt} + \gamma_1 w_t + \gamma_2 w.$$

Here, the first term is the contribution of the inertia of the container, and the second term represents the damping that the granular material provides, which is assumed to be proportional to the velocity, and so γ_1 is the damping coefficient. Thus, the force balance at the end $x = l$ is

$$\rho_3 w_{tt} + \gamma_1 w_t + \gamma_2 w + \alpha_2 v_x(l,.) = 0 \qquad \text{in} \qquad]0,+\infty[, \tag{1.5}$$

where the parameters γ_1 and γ_2 are non-negative constants. Finally, the initial data are given by

$$u(0) = u_0, \quad u_t(0) = u_1 \qquad \text{in} \quad]0,l_0[, \tag{1.6}$$

$$v(0) = v_0, \quad v_t(0) = v_1 \qquad \text{in} \quad]l_0,l[, \tag{1.7}$$

$$w(0) = w_0 \in \mathbb{C}, \qquad w_t(0) = w_1 \in \mathbb{C}. \tag{1.8}$$

Here, we assume the following hypotheses on the relaxation function g:

$$g(t) \geq 0, \quad \forall t \geq 0, \quad \text{and} \quad g > 0 \text{ almost everywhere in } [0,+\infty[; \tag{1.9}$$

$$\exists\, k_1,k_2 > 0: \quad -k_1 g(t) \leq g'(t) \leq -k_2 g(t), \quad \forall t \geq 0; \tag{1.10}$$

$$0 < \alpha := \alpha_1 - \int_0^\infty g(s)ds. \tag{1.11}$$

Concerning models of motion with the attached tip body, Andrews and Shillor [1] establish the existence and uniqueness of the model and showed the exponential energy decay of the solution provided and extra damping term is present. In [11] Zietsman, Rensburg and Merwe consider the effect of boundary damping on a cantilevered Timoshenko beam with a rigid body attached to the free end. The authors establish the efficiency and accuracy of the finite element method for calculating the eigenvalues and eigenmodes. Although no conclusion is showed with regard to the stabilization of the system, the authors showed interesting phenomena concerning the damped vibration spectrum and the associated eigenmodes. See also the work of Feireisl and O'Dowd [7] where is showed, for an hybrid system composed of a cable with masses at both end, the strong stability for a nonlinear and nonmonotone feedback law applied at one end.

The main result of this paper is to show that the system (1.1)–(1.8) is exponentially stable if and only if the memory effect is effective over the viscoelastic part of the material. This means that the dissipative properties given by the tip load is not enough to produce exponencial rate of decay when the memory effect is not effective. Finally, when $g = 0$, we prove that the system is not exponentially stable but the dissipation given by the tip load produce polynomial stability. The method we use is based on Prüss Theorem to show exponential stability. The proof of the lack of exponential stability is based on the Weyl invariance Theorem and the proof of the polinomial stability is based on the Borichev and Tomilov result.

2 EXISTENCE AND UNIQUENESS OF SOLUTIONS

To use the semigroup approach we need to rewrite the problem as an autonomous system. For this reason we introduce the *history problem*, obtained by replacing the equation (1.1) by the following history equation

$$\rho_1 u_{tt} - \alpha_1 u_{xx} + \int_{-\infty}^t g(t-s)u_{xx}(.,s)ds = 0 \qquad \text{in} \qquad]0,l_0[\times]0,+\infty[.$$

Following the ideas of Dafermos [4], [3] and Fabrizio [6], we introduce the notation

$$\eta(x,t,s) := u(x,t) - u(x,t-s),$$

with $s \in [0,+\infty)$; whence we consider the system

$$\rho_1 u_{tt} - \alpha u_{xx} - \int_0^\infty g(s)\eta_{xx}(s)ds = 0 \quad \text{in} \quad]0,l_0[\times]0,+\infty[\tag{2.1}$$

$$\rho_2 v_{tt} - \alpha_2 v_{xx} = 0 \quad \text{in} \quad]l_0,l[\times]0,+\infty[\tag{2.2}$$

$$\eta_t + \eta_s - u_t = 0 \quad \text{in} \quad]0,l_0[\times]0,+\infty[\times]0,+\infty[. \tag{2.3}$$

with u, v and w, satisfying (1.5) and the initial conditions (1.6), (1.7), (1.8) and η verifying

$$\eta(x,0,s) = \eta_0(x,s) =: u_0(x) - u_0(x,-s), \qquad \forall\,(x,s) \in\ ]0,l_0[\times]0,+\infty[, \tag{2.4}$$

with boundary conditions are given by

$$\eta(x,t,0) = 0, \qquad \forall\,(x,t) \in]0,l_0[\times]0,+\infty[, \tag{2.5}$$

$$\eta(0,t,s) = 0, \qquad \forall\,(t,s) \in]0,+\infty[\times]0,+\infty[. \tag{2.6}$$

The transmission conditions now are given by

$$u(l_0,t) = v(l_0,t), \quad \alpha u_x(l_0,t) + \int_0^\infty g(s)\eta_x(l_0,t,s)ds = \alpha_2 v_x(l_0,t), \quad \forall t \geq 0. \tag{2.7}$$

We define the total energy of the system as

$$\begin{aligned} E(t) &= \frac{1}{2}\left\{\int_0^{l_0}\left[\rho_1|u_t|^2 + \alpha|u_x|^2 + \int_0^\infty g(s)|\eta_x(s)|^2 ds\right]dx + \right. \\ &\quad \left. + \int_{l_0}^{l}\left[\rho_2|v_t|^2 + \alpha_2|v_x|^2\right]dx + \rho_3|w_t|^2 + \gamma_2|w|^2\right\}. \end{aligned}$$

Let us introduce the following spaces:

$$\mathbb{H}^m := H^m(0,l_0)\times H^m(l_0,l), \quad m\in\mathbb{N};$$

$$\mathbb{H}^m_* := \{(u,v)\in\mathbb{H}^m;\ u(0)=0,\ u(l_0)=v(l_0)\},\ \ m\in\mathbb{N};$$

$$\mathbb{L}^2 := L^2(0,l_0)\times L^2(l_0,l);$$

$$H^m_*(0,l_0) := \{f\in H^m(0,l_0);\ f(0)=0\},\ \ m\in\mathbb{N};$$

$$L^2_g := \left\{\varphi:\mathbb{R}^+\to H^1_*(0,l_0);\ \ \int_0^\infty g(s)\int_0^{l_0}|\varphi_x(s)|^2 dxds < \infty\right\}.$$

We recall that L^2_g is a Hilbert space when endowed with the inner product given by

$$\langle\varphi,\psi\rangle_{L^2_g} = \int_0^\infty g(s)\int_0^{l_0}\varphi_x(s)\overline{\psi_x(s)}dxds.$$

With this notations, we consider the phase space

$$\mathscr{H} := \left\{(u,v,U,V,\eta,w,W)^T\in\mathbb{H}^1_*\times\mathbb{L}^2\times L^2_g\times\mathbb{C}^2;\ \ v(l)=w\right\}.$$

Note that the space $\mathscr{H}$ is a Hilbert space with the norm

$$\begin{aligned}\|\mathscr{U}\|_{\mathscr{H}}^2 = \ & \alpha\|u_x\|_{L^2(0,l_0)}^2 + \alpha_2\|v_x\|_{L^2(0,l_0)}^2 + \rho_1\|U\|_{L^2(0,l_0)}^2 + \\ & + \rho_2\|V\|_{L^2(0,l_0)}^2 + \|\eta\|_{L_g^2}^2 + \gamma_2|w|^2 + \rho_3|W|^2.\end{aligned}$$

where $\mathscr{U} = (u,v,U,V,\eta,w,W)^T \in \mathscr{H}$.

Let us introduce the linear unbounded operator $\mathscr{A}$ in $\mathscr{H}$ as follows

$$\mathscr{A}\,\mathscr{U} = \begin{pmatrix} U \\ V \\ \dfrac{\alpha}{\rho_1}u_{xx} + \dfrac{1}{\rho_1}\displaystyle\int_0^\infty g(s)\eta_{xx}(s)ds \\ \dfrac{\alpha_2}{\rho_2}v_{xx} \\ U - \eta_s \\ W \\ -\dfrac{\gamma_1}{\rho_3}W - \dfrac{\gamma_2}{\rho_3}w - \dfrac{\alpha_2}{\rho_3}v_x(l) \end{pmatrix},$$

with domain

$$D(\mathscr{A}) = \left\{ \begin{array}{c} \mathscr{U} = (u,v,U,V,\eta,w,W)^T \in \mathscr{H};\left(\alpha u + \displaystyle\int_0^\infty g(s)\eta(s)ds, v\right) \in \mathbb{H}^2, (U,V) \in \mathbb{H}_*^1, \\ V(l) = W,\ \eta|_{s=0} = 0,\ \eta_s \in L_g^2,\ \alpha u_x(l_0) + \displaystyle\int_0^\infty g(s)\eta_x(l_0,s)ds = \alpha_2 v_x(l_0) \end{array} \right\}.$$

Using the hypotheses on g, a direct computation yields

$$\text{Re}\,\langle \mathscr{A}\,\mathscr{U}, \mathscr{U}\rangle = -\gamma_1|W|^2 + \frac{1}{2}\int_0^{l_0}\int_0^\infty g'(s)|\eta_x(s)|^2 ds dx \ \le \ 0,$$

which means that $\mathscr{A}$ is a dissipative operator. The system (2.1)-(2.7), (1.5)-(1.8) is equivalent to

$$\mathscr{U}_t = \mathscr{A}\,\mathscr{U}, \qquad \mathscr{U}(0) = \mathscr{U}_0; \tag{2.8}$$

where
$\mathscr{U}(t) = (u(t),v(t),U(t),V(t),\eta(t),w(t),W(t))^T$ and $\mathscr{U}_0 = (u_0,v_0,u_1,v_1,\eta_0,w_0,w_1)^T$.

Under this conditions, we have

Theorem 2.1. *The operator $\mathscr{A}$ is the infinitesimal generator of a C_0-semigroup of contractions $(S(t))_{t\geq 0}$ on $\mathscr{H}$. Thus, for any initial data $\mathscr{U}_0 \in \mathscr{H}$, the problem (2.8) has a unique weak (mild) solution*

$$\mathscr{U} \in \mathscr{C}^0([0,\infty[,\mathscr{H}).$$

Moreover, if $\mathscr{U}_0 \in D(\mathscr{A})$, then $\mathscr{U}$ is a strong solution of (2.8), that is

$$\mathscr{U} \in \mathscr{C}^1([0,\infty[,\mathscr{H}) \cap \mathscr{C}^0([0,\infty[,D(\mathscr{A})).$$

Proof. It easy to see that $D(\mathscr{A})$ is dense in $\mathscr{H}$; and, since $\mathscr{A}$ is a dissipative operator, it is enought to show that $0 \in \rho(\mathscr{A})$. To do that, we will show that for $F = (f^1, f^2, \cdots, f^7)^T \in \mathscr{H}$, there exists only one $\mathscr{U} = (u, v, U, V, \eta, w, W)^T \in D(\mathscr{A})$ such that $\mathscr{A}\ \mathscr{U} = F$. In terms of the components, we have

$$U = f^1 \tag{2.9}$$

$$V = f^2 \tag{2.10}$$

$$\frac{\alpha}{\rho_1}u_{xx} + \frac{1}{\rho_1}\int_0^\infty g(s)\eta_{xx}(s)ds = f^3 \tag{2.11}$$

$$\frac{\alpha_2}{\rho_2}v_{xx} = f^4 \tag{2.12}$$

$$U - \eta_s = f^5 \tag{2.13}$$

$$W = f^6 = f^2(l) \tag{2.14}$$

$$-\frac{\gamma_1}{\rho_3}W - \frac{\gamma_2}{\rho_3}w - \frac{\alpha_2}{\rho_3}v_x(l) = f^7 \tag{2.15}$$

Indeed, from the equations (2.9) and (2.13), we get that $\eta_s \in L^2_g$ and that

$$\eta(x,s) = sf^1(x) - \int_0^s f^5(x,\tau)d\tau$$

which means that η is uniquely determined. Moreover, using (1.10) and (2.5), we can write, for each $T > 0$:

$$\begin{aligned}
\int_0^T g(s)\int_0^{l_0}|\eta_x(s)|^2dxds &\leq \frac{2}{k_2}\int_0^T g(s)\int_0^{l_0}\eta_x(s)\overline{\eta_{sx}(s)}dxds \\
&\leq \frac{1}{2}\int_0^T g(s)\int_0^{l_0}|\eta_x(s)|^2dxds + \frac{2}{k_2^2}\int_0^T g(s)\int_0^{l_0}|\eta_{sx}(s)|^2dxds
\end{aligned}$$

from we obtain

$$\|\eta\|_{L^2_g} \leq \frac{2}{k_2}\|\eta_s\|_{L^2_g}$$

which enables us to conclude that $\eta \in L^2_g$. Thus, it remains only to establish the existence and uniqueness of solution for the system

$$(P)\begin{cases}
u_{xx} = \dfrac{\rho_1}{\alpha}f^3 - \dfrac{1}{\alpha}\displaystyle\int_0^\infty g(s)\eta_{xx}(s)ds \\
v_{xx} = \dfrac{\rho_2}{\alpha_2}f^4 \\
u(0) = 0, \quad u(l_0) = v(l_0), \quad \alpha_2 v_x(l) + \gamma_2 v(l) = -\rho_3 f^7 - \gamma_1 f^6 \\
\alpha u_x(l_0) - \alpha_2 v_x(l_0) = -\displaystyle\int_0^\infty g(s)\eta_x(l_0,s)ds.
\end{cases}$$

Let us consider the functional $T : X \to \mathbb{C}$ given by

$$T(\varphi,\psi) := \quad -\rho_1\int_0^{l_0}\overline{f^3}\varphi dx - \int_0^{l_0}\left(\overline{\int_0^{\infty} g(s)\eta_x(s)ds}\right)\varphi_x dx + \\ +\rho_2\int_{l_0}^{l}\left(\overline{\int_{l_0}^{x} f^4(\tau)d\tau}\right)\psi_x dx + \overline{\tilde{G}}\psi(l)$$

for all $(\varphi,\psi)\in X$, where $\tilde{G} := \left(G - \rho_2\int_{l_0}^{l} f^4 dx\right)$, and $X := \mathbb{H}^1_*$ is a Hilbert space, endowed with the inner product

$$\langle(\varphi,\psi),(u,v)\rangle_X = \alpha\int_0^{l_0}\overline{u_x}\varphi_x dx + \alpha_2\int_{l_0}^{l}\overline{v_x}\psi_x dx + \gamma_2\overline{v(l)}\psi(l).$$

It's clear that $T\in X'$; therefore, by the Riesz representation theorem we conclude that there exists only one weak solution to system (P). So we have that $0\in\rho(\mathscr{A})$. □

3 EXPONENTIAL STABILITY

In this section, we show that if hypothesis (1.9)–(1.11) hold, then the corresponding semigroup is exponentially stable. The main tool we use is Prüss's results [9], which is summarized in the following theorem.

Theorem 3.2. *Let $(\mathscr{S}(t))_{t\geqslant 0}$ be a C_0-semigroup on a Hilbert space $\mathscr{H}$ generated by $\mathscr{A}$. Then the semigroup is exponentially stable if and only if*

$$i\mathbb{R}\subset\rho(\mathscr{A}), \quad \text{and} \quad \|(i\lambda I-\mathscr{A})^{-1}\|_{\mathscr{L}(\mathscr{H})}\leqslant C, \qquad \forall\lambda\in\mathbb{R}.$$

In the next Lemma we show that the imaginary axis is contained in the resolvent set.

Lemma 3.1. *Under the hypotheses (1.9)-(1.11), the operator $\mathscr{A}$ verify*

$$i\mathbb{R}\subset\rho(\mathscr{A}). \tag{3.1}$$

Proof. In the Theorem 2.1, we have already shown that $0\in\rho(\mathscr{A})$. Moreover, note that we can't conclude that the spectrum of $\mathscr{A}$ is formed only by eigenvalues, since $\mathscr{A}^{-1}$ is not compact. So, if (3.1) is not true, then there exists $\lambda_0\in\mathbb{R}$ with $\|\mathscr{A}^{-1}\|^{-1}\leq|\lambda_0|$, such that $\{i\lambda;|\lambda|<|\lambda_0|\}\subset\rho(\mathscr{A})$ and $\sup\{\|(i\lambda-\mathscr{A})^{-1}\|;|\lambda|<|\lambda_0|\}=\infty$. Follow that there exist sequences $(\lambda_n)_n\subset\mathbb{R}$ and $(\mathscr{U}_n)_n=((u_n,v_n,U_n,V_n,\eta_n,w_n,W_n)^T)_n\subset D(\mathscr{A})$, such that

$$\lambda_n \longrightarrow |\lambda_0|, \tag{3.2}$$

$$\|\mathscr{U}_n\|_{\mathscr{H}}=1, \quad \forall\ n\in\mathbb{N}, \tag{3.3}$$

$$(i\lambda_n-\mathscr{A})\mathscr{U}_n=F_n=(f_n^1,...,f_n^7)\longrightarrow 0 \quad \text{in } \mathscr{H}. \tag{3.4}$$

But, (3.4) is equivalent to

$$i\lambda_n u_n - U_n = f_n^1 \longrightarrow 0 \quad \text{in } H^1(0,l_0) \tag{3.5}$$

$$i\lambda_n v_n - V_n = f_n^2 \longrightarrow 0 \quad \text{in } H^1(l_0,l) \tag{3.6}$$

$$i\lambda_n U_n - \frac{\alpha}{\rho_1} u_{n,xx} - \frac{1}{\rho_1}\int_0^\infty g(s)\eta_{n,xx}(s)ds = f_n^3 \longrightarrow 0 \quad \text{in } L^2(0,l_0) \tag{3.7}$$

$$i\lambda_n V_n - \frac{\alpha_2}{\rho_2} v_{n,xx} = f_n^4 \longrightarrow 0 \quad \text{in } L^2(l_0,l) \tag{3.8}$$

$$i\lambda_n \eta_n - U_n + \eta_{n,s} = f_n^5 \longrightarrow 0 \quad \text{in } L_g^2 \tag{3.9}$$

$$i\lambda_n w_n - W_n = f_n^6 \longrightarrow 0 \quad \text{in } \mathbb{C} \tag{3.10}$$

$$i\lambda_n W_n + \frac{\gamma_1}{\rho_3} W_n + \frac{\gamma_2}{\rho_3} w_n + \frac{\alpha_2}{\rho_3} v_{n,x}(l) = f_n^7 \longrightarrow 0 \quad \text{in } \mathbb{C} \tag{3.11}$$

Taking the inner product of (3.4) with $\mathscr{U}_n$ in $\mathscr{H}$, we get

$$\text{Re}\,\langle \mathscr{A}\,\mathscr{U}_n, \mathscr{U}_n\rangle = -\gamma_1 |W_n|^2 + \frac{1}{2}\int_0^\infty g'(s)|\eta_{n,x}(s)|^2 dsdx \longrightarrow 0. \tag{3.12}$$

Follows from condition (1.10) and (3.12) that

$$\sqrt{\gamma_1}|W_n| \longrightarrow 0 \qquad \text{and} \qquad \|\eta_n\|_{L_g^2} \longrightarrow 0. \tag{3.13}$$

Now, we use (3.3) and (3.5) for conclude that there exist $U, u \in L^2(0,l_0)$ and subsequences still denoted by $(U_n)_n$ and $(u_n)_n$, such that

$$U_n \longrightarrow U \quad \text{in } L^2(0,l_0) \qquad \text{and} \qquad u_n \longrightarrow u \quad \text{in } L^2(0,l_0). \tag{3.14}$$

Moreover, using (3.3), (3.7), (3.13) and (3.14), we get

$$u_{n,x} \longrightarrow u_x \quad \text{in } L^2(0,l_0), \tag{3.15}$$

$$\alpha u_{n,xx} + \int_0^\infty g(s)\eta_{n,xx}(s)ds \longrightarrow \alpha u_{xx} \quad \text{in } L^2(0,l_0).$$

So, from (3.14), (3.15) and (3.5), we can conclude that

$$u_n \longrightarrow u \quad \text{in } H^1(0,l_0) \qquad \text{and} \qquad U_n \longrightarrow U \quad \text{in } H^1(0,l_0).$$

Proceeding analogously, we find

$$v_n \longrightarrow v \quad \text{in } H^2(l_0,l) \qquad \text{and} \qquad V_n \longrightarrow V \quad \text{in } H^1(l_0,l).$$

From convergences above, remembering $w_n = v_n(l)$ and $W_n = V_n(l)$, follow that

$$w_n \longrightarrow w := v(l), \qquad W_n \longrightarrow W := V(l), \qquad \text{and} \qquad v_{n,x}(l) \longrightarrow v_x(l).$$

The convergences obtained above allow us to pass to the limit in (3.5)-(3.11), obtaining the following system

$$
\begin{aligned}
i|\lambda_0|u - U &= 0\\
i|\lambda_0|v - V &= 0\\
i|\lambda_0|\rho_1 U - \alpha u_{xx} &= 0\\
i|\lambda_0|\rho_2 V - \alpha_2 v_{xx} &= 0\\
i|\lambda_0|w - W &= 0\\
i|\lambda_0|W + \frac{\gamma_1}{\rho_3}W + \frac{\gamma_2}{\rho_3}w + \frac{\alpha_2}{\rho_3}v_x(l) &= 0
\end{aligned}
$$

We conclude that there exists $\mathscr{U} = (u, v, U, V, 0, v(l), V(l))^T \in D(\mathscr{A})$, such that

$$\mathscr{U}_n \longrightarrow \mathscr{U} \quad \text{in } \mathscr{H}, \tag{3.16}$$

where (u, v) is precisely the solution of the system given by

$$
\begin{cases}
u_{xx} + \dfrac{\rho_1|\lambda_0|^2}{\alpha}u = 0 & \text{in} \quad]0, l_0[\\[2mm]
v_{xx} + \dfrac{\rho_2|\lambda_0|^2}{\alpha_2}v = 0 & \text{in} \quad]l_0, l[\\[2mm]
u(0) = 0, \qquad u(l_0) = v(l_0) & \\[2mm]
\alpha u_x(l_0) = \alpha_2 v_x(l_0), \qquad v(l) = 0 &
\end{cases}
$$

when $\gamma_1 > 0$; or, in the case that $\gamma_1 = 0$, of the system obtained in the above system, replacing the boundary condition $v(l) = 0$ by $v_x(l) + \alpha_3 v(l) = 0$, where $\alpha_3 = \alpha_2^{-1}(\gamma_2 - |\lambda_0|^2\rho_3)$.

However, each one of these two systems has a unique solution, namely, the null solution; from which it follows that $\mathscr{U} = 0$. With this, we rewrite (3.16) as

$$\mathscr{U}_n \longrightarrow 0 \quad \text{in } \mathscr{H}, \tag{3.17}$$

which contradicts (3.3), completing the proof. □

From now, on C will denote a generic constant, that can be different in different places. Let us denote by

$$b := \int_0^\infty g(s)ds. \tag{3.18}$$

We will prove that the solution $\mathscr{U}$ of the resolvent equation

$$(i\lambda I - \mathscr{A})\mathscr{U} = F \tag{3.19}$$

is uniformly bounded for any take $F = (f^1, f^2, ..., f^7)^T \in \mathscr{H}$, that is,

$$\|\mathscr{U}\|_{\mathscr{H}} \leq C\|F\|_{\mathscr{H}} \quad \forall F \in \mathscr{H}.$$

In fact, in terms of the components we have

$$i\lambda u - U = f^1 \tag{3.20}$$
$$i\lambda v - V = f^2 \tag{3.21}$$
$$i\lambda\rho_1 U - \alpha u_{xx} - \int_0^\infty g(s)\eta_{xx}(s)ds = \rho_1 f^3 \tag{3.22}$$
$$i\lambda\rho_2 V - \alpha_2 v_{xx} = \rho_2 f^4 \tag{3.23}$$
$$i\lambda\eta - U + \eta_s = f^5 \tag{3.24}$$
$$i\lambda w - W = f^6 \tag{3.25}$$
$$i\lambda\rho_3 W + \gamma_1 W + \gamma_2 w + \alpha_2 v_x(l) = \rho_3 f^7 \tag{3.26}$$

The dissipative properties of $\mathscr{A}$ implies that there exists a positive constant C such that

$$\gamma_1|W|^2 + \int_0^{l_0}\int_0^\infty g(s)|\eta_x(s)|^2 dsdx \le C\|\mathscr{U}\|_{\mathscr{H}}\|F\|_{\mathscr{H}}. \tag{3.27}$$

The following Lemma will play an important role in the sequel.

Lemma 3.2. *Under the above notations, for any $\varepsilon > 0$ sufficiently small, there exist a constant $C_\varepsilon > 0$ such that, for $|\lambda|$ large enough, hold*

$$\int_0^{l_0} |U|^2 + |u_x|^2 dx \le C_\varepsilon\|\mathscr{U}\|_{\mathscr{H}}\|F\|_{\mathscr{H}} + C_\varepsilon\|F\|^2_{\mathscr{H}} + \varepsilon\left|\alpha u_x(l_0) + \int_0^\infty g(s)\eta_x(l_0,s)ds\right|^2.$$

Proof. Multiplying the equation (3.22) by $\int_0^\infty g(s)\overline{\eta(s)}ds$ and using (3.24), we get

$$\begin{aligned} b\rho_1\int_0^{l_0}|U|^2dx &= \rho_1\int_0^{l_0}\int_0^\infty g(s)\overline{\eta_s(s)}U dsdx - \rho_1\int_0^{l_0}\int_0^\infty g(s)\overline{f^5}U dsdx + \\ &+\alpha\int_0^{l_0}\int_0^\infty g(s)\overline{\eta_x(s)}u_x dsdx + \int_0^{l_0}\left|\int_0^\infty g(s)\eta_x(s)ds\right|^2 dx - \rho_1\int_0^{l_0}\int_0^\infty g(s)\overline{\eta(s)}f^3 dsdx + \\ &-\underbrace{\left[\alpha u_x(l_0) + \int_0^\infty g(s)\eta_x(l_0,s)ds\right]\left[\int_0^\infty g(s)\overline{\eta(l_0,s)}ds\right]}_{:=R_1}. \end{aligned}$$

For each $\varepsilon > 0$, we use (1.10) and (3.27) for obtain

$$\begin{aligned} \operatorname{Re}\left[\int_0^{l_0}\int_0^\infty g(s)\overline{\eta_s(s)}U dsdx\right] &\le \varepsilon\|U\|^2_{L^2(0,l_0)} + C_\varepsilon\int_0^{l_0}\int_0^\infty g(s)|\eta_x(s)|^2ds \\ &\le \varepsilon\|U\|^2_{L^2(0,l_0)} + C_\varepsilon\|\mathscr{U}\|_{\mathscr{H}}\|F\|_{\mathscr{H}}. \end{aligned}$$

Using (3.27) once more we get

$$\operatorname{Re}\left[\alpha\int_0^{l_0}\int_0^\infty g(s)\overline{\eta_x(s)}u_x dsdx\right] \le \varepsilon\|u_x\|^2_{L^2(0,l_0)} + C_\varepsilon\|\mathscr{U}\|_{\mathscr{H}}\|F\|_{\mathscr{H}}$$

and

$$|R_1| \leq C_\varepsilon \|\mathscr{U}\|_{\mathscr{H}} \|F\|_{\mathscr{H}} + \varepsilon \left| \alpha u_x(l_0) + \int_0^\infty g(s)\eta_x(l_0,s)ds \right|^2.$$

Therefore, for $\varepsilon > 0$ sufficiently small, we have

$$\int_0^{l_0} |U|^2 dx \leq C_\varepsilon \|\mathscr{U}\|_{\mathscr{H}} \|F\|_{\mathscr{H}} + \varepsilon \int_0^{l_0} |u_x|^2 dx + \varepsilon \left| \alpha u_x(l_0) + \int_0^\infty g(s)\eta_x(l_0,s)ds \right|^2. \tag{3.28}$$

On the other hand, multiplying (3.22) by $\overline{u}$ and using (3.20), we get

$$\alpha \int_0^{l_0} |u_x|^2 dx = \rho_1 \int_0^{l_0} |U|^2 dx + \rho_1 \int_0^{l_0} U\overline{f^1} dx + \rho_1 \int_0^{l_0} f^3 \overline{u} dx + \\ - \int_0^{l_0}\int_0^\infty g(s)\eta_x(s)\overline{u_x} ds dx + \underbrace{\left(\alpha u_x(l_0) + \int_0^\infty g(s)\eta_x(l_0,s)ds \right) \overline{u(l_0)}}_{:=R_2}.$$

Since

$$|u(l_0)| \leq \|u\|_{L^\infty} \leq C\|u\|_{L^2}^{1/2} \|u_x\|_{L^2}^{1/2},$$

using (3.20) we get, for each $\varepsilon > 0$ and for $\lambda \neq 0$:

$$\begin{aligned} |R_2| &\leq \varepsilon \left| \alpha u_x(l_0) + \int_0^\infty g(s)\eta_x(l_0,s)ds \right|^2 + C_\varepsilon |u(l_0)|^2 \\ &\leq \varepsilon \left| \alpha u_x(l_0) + \int_0^\infty g(s)\eta_x(l_0,s)ds \right|^2 + \varepsilon \|u_x\|_{L^2}^2 + C_\varepsilon \|u\|_{L^2}^2 \\ &\leq \varepsilon \left| \alpha u_x(l_0) + \int_0^\infty g(s)\eta_x(l_0,s)ds \right|^2 + \varepsilon \|u_x\|_{L^2}^2 + \frac{C_\varepsilon}{|\lambda|} \|U\|_{L^2}^2 + \frac{C_\varepsilon}{|\lambda|} \|F\|_{\mathscr{H}}^2. \end{aligned}$$

Therefore, for $\varepsilon > 0$ sufficiently small, we have

$$\begin{aligned} \int_0^{l_0} |u_x|^2 dx \leq \quad & C_{\varepsilon,\lambda} \int_0^{l_0} |U|^2 dx + C_\varepsilon \|\mathscr{U}\|_{\mathscr{H}} \|F\|_{\mathscr{H}} + C_{\varepsilon,\lambda} \|F\|_{\mathscr{H}}^2 + \\ & + \varepsilon \left| \alpha u_x(l_0) + \int_0^\infty g(s)\eta_x(l_0,s)ds \right|^2. \end{aligned}$$

From above inequality and (3.28), our conclusion follows. □

The next lemma is crucial to ensure that the exponential decay occurs in the case where $\gamma_1 = 0$. Indeed, it provides an estimate for the term involving $|W|^2$ that can be obtained from (3.27) only when γ_1 is positive.

Lemma 3.3. *There exist $C > 0$ such that*

$$\rho_2 |W|^2 \leq C \|\mathscr{U}\|_{\mathscr{H}} \|F\|_{\mathscr{H}} + C \int_0^{l_0} \alpha |v_x|^2 dx.$$

Proof. Multiplying (3.23) by $(x-l_0)\overline{v_x}$, using (3.21), and remembering that $V(l)=W$, we get

$$\begin{aligned}\rho_2|W|^2+\alpha_2|v_x(l)|^2 = & -\frac{\rho_2}{l-l_0}\int_{l_0}^{l}|V|^2dx-\frac{2\rho_2}{l-l_0}\int_{l_0}^{l}(x-l_0)V\overline{f_x^2}dx+ \\ & +\frac{\alpha_2}{l-l_0}\int_{l_0}^{l}|v_x|^2dx-\frac{2\rho_2}{l-l_0}\int_{l_0}^{l}(x-l_0)f^4\overline{v_x}dx.\end{aligned}$$

Taking the real part, we get the desired inequality. □

Now we are in condition to show the main result of this section.

Theorem 3.3. *Let us suppose that (1.9)-(1.11) hold. Then the semigroup $e^{\mathscr{A}t}$ is exponentially stable.*

Proof. In view of Proposition 1, we only need to show that there exist $C>0$ such that:

$$\|(i\lambda I-\mathscr{A})^{-1}\|_{\mathscr{L}(\mathscr{H})}\leq C,\quad \forall\lambda\in\mathbb{R}.$$

Since the resolvent operator is holomorphic, it is enough to prove the above inequality for $|\lambda|$ large enough. In fact, multiplying (3.23) by $(l-x)\overline{v_x}$ and using (3.21), we get

$$\begin{aligned}\int_{l_0}^{l}\left[\rho_2|V|^2+\alpha_2|v_x|^2\right]dx = \quad & (l-l_0)\left[\rho_2|V(l_0)|^2+\alpha_2|v_x(l_0)|^2\right]+ \\ & -2\rho_2\int_{l_0}^{l}(l-x)V\overline{f_x^2}dx-2\rho_2\int_{l_0}^{l}(l-x)f^4\overline{v_x}dx.\end{aligned}$$

Taking the real part, we get

$$\int_{l_0}^{l}\left[\rho_2|V|^2+\alpha_2|v_x|^2\right]dx\leq C\left[\alpha_2|v_x(l_0)|^2+\rho_2|V(l_0)|^2\right]+C\|\mathscr{U}\|_{\mathscr{H}}\|F\|_{\mathscr{H}}. \tag{3.29}$$

On the other hand, multiplying (3.22) by $x\left(\overline{\alpha u_x+\int_0^\infty g(s)\eta_x(s)ds}\right)$, and using (3.20) and (3.24), we get

$$\rho_1|U(l_0)|^2+\left|\alpha u_x(l_0)+\int_0^\infty g(s)\eta_x(l_0,s)ds\right|^2\leq C\int_0^{l_0}\left[\alpha|u_x|^2+\rho_1|U|^2\right]dx+C\|\mathscr{U}\|_{\mathscr{H}}\|F\|_{\mathscr{H}}.$$

From Lemma 3.2, we get, for ε small enough and for $|\lambda|$ large enough, that

$$\rho_1|U(l_0)|^2+\left|\alpha u_x(l_0)+\int_0^\infty g(s)\eta_x(l_0,s)ds\right|^2\leq C\|\mathscr{U}\|_{\mathscr{H}}\|F\|_{\mathscr{H}}+C\|F\|^2_{\mathscr{H}}. \tag{3.30}$$

Using the transmission conditions, inequality (3.29) can be estimated by (3.30), that is

$$\int_{l_0}^{l}\left[\alpha_2|v_x|^2+\rho_2|V|^2\right]dx\leq C\|\mathscr{U}\|_{\mathscr{H}}\|F\|_{\mathscr{H}}+C\|F\|^2_{\mathscr{H}}. \tag{3.31}$$

Moreover from Lemma 3.2 and inequality (3.30), we get

$$\int_0^{l_0} \left[\rho_1|U|^2+\alpha|u_x|^2\right]dx \;\leq\; C\|\mathscr{U}\|_{\mathscr{H}}\|F\|_{\mathscr{H}}+C\|F\|_{\mathscr{H}}^2.$$

Therefore, Lemma 3.3, equation (3.25), and inequality (3.31) implies

$$|W|^2+|w|^2 \leq C\|\mathscr{U}\|_{\mathscr{H}}\|F\|_{\mathscr{H}}+C\|F\|_{\mathscr{H}}^2.$$

From the three last inequalities and (3.27), we get

$$\|\mathscr{U}\|_{\mathscr{H}}^2 \leq C\|\mathscr{U}\|_{\mathscr{H}}\|F\|_{\mathscr{H}}+C\|F\|_{\mathscr{H}}^2,$$

which implies in

$$\|(i\lambda I-\mathscr{A})^{-1}F\|_{\mathscr{H}} \;=\; \|\mathscr{U}\|_{\mathscr{H}} \;\leq\; C\|F\|_{\mathscr{H}}.$$

Then, the semigroup is exponentially stable. □

4 THE LACK OF EXPONENTIAL STABILITY

Now we shall prove that the dissipation given by the memory effect is necessary for exponential stability of the system. Let us consider the *problem without memory effect*; namely

$$\rho_1 u_{tt}-\alpha_1 u_{xx} = 0 \quad \text{in} \quad]0,l_0[\times]0,+\infty[\tag{4.1}$$

$$\rho_2 v_{tt}-\alpha_2 v_{xx} = 0 \quad \text{in} \quad]l_0,l[\times]0,+\infty[\tag{4.2}$$

$$\rho_3 w_{tt}+\gamma_1 w_t+\gamma_2 w+\alpha_2 v_x(l) = 0 \quad \text{in} \quad]0,+\infty[\tag{4.3}$$

with boundary conditions

$$u(0)=0, \quad v(l)=w \quad \text{in} \quad]0,+\infty[\tag{4.4}$$

and with transmission conditions

$$u(l_0)=v(l_0), \quad \alpha_1 u_x(l_0)=\alpha_2 v_x(l_0) \quad \text{in} \quad]0,+\infty[\tag{4.5}$$

and initial data

$$(u(0),v(0),u_t(0),v_t(0),w(0),w_t(0))=(u_0,v_0,u_1,v_1,w_0,w_1)\in\breve{\mathscr{H}}, \tag{4.6}$$

where α_1, α_2, ρ_1, ρ_2, ρ_3, γ_2 are as before, and γ_1, now, is a positive constant. Moreover, for this problem, we consider the phase space

$$\breve{\mathscr{H}}=\left\{\mathscr{U}=(u,v,U,V,w,W)^T\in\mathbb{H}_*^1\times\mathbb{L}^2\times\mathbb{C}^2;\; v(l)=w\right\}.$$

The total energy associed with the system is

$$\breve{E}(t)=\frac{1}{2}\left[\int_0^{l_0}\left[\rho_1|u_t|^2+\alpha_1|u_x|^2\right]dx+\int_{l_0}^{l}\left[\rho_2|v_t|^2+\alpha_2|v_x|^2\right]dx+\rho_3|w_t|^2+\gamma_2|w|^2\right]$$

and it is not difficult to see that, for all $\mathscr{U} \in \breve{\mathscr{H}}$, we have

$$\frac{d}{dt}\breve{E}(t) = -\gamma_1 |w_t|^2.$$

Let us denote by $\mathscr{B}$ the unbounded operator of $\breve{\mathscr{H}}$ given by

$$\mathscr{B}\,\mathscr{U} = \begin{pmatrix} U \\ V \\ \frac{\alpha_1}{\rho_1} u_{xx} \\ \frac{\alpha_2}{\rho_2} v_{xx} \\ W \\ -\frac{\gamma_1}{\rho_3} W - \frac{\gamma_2}{\rho_3} w - \frac{\alpha_2}{\rho_3} v_x(l) \end{pmatrix}$$

with domain

$$D(\mathscr{B}) = \left\{ \mathscr{U} = (u,v,U,V,w,W)^T \in (\mathbb{H}^1_* \cap \mathbb{H}^2) \times \mathbb{H}^1_* \times \mathbb{C}^2; V(l) = W, \alpha_1 u_x(l_0) = \alpha_2 v_x(l_0) \right\}.$$

It is easy to see that

$$\mathrm{Re}(\mathscr{B}\mathscr{U}, \mathscr{U})_{\breve{\mathscr{H}}} = -\gamma_1 |W|^2. \tag{4.7}$$

It is not difficult to see that the operator $\mathscr{B}$ is the infinitesimal generator of a C_0-semigroup of contractions over $\breve{\mathscr{H}}$, which we will denote by $T(t)$. This shows that the problem (4.1)-(4.6) is well-posed.

To prove that the system (4.1)-(4.6) is not exponentially stable, the main tool we use is the Weyl's theorem about the invariance of the essential spectral radius by compact perturbations. To do that, let us consider the following *conservative system*

$$\rho_1 \tilde{u}_{tt} - \alpha_1 \tilde{u}_{xx} = 0 \quad \text{in} \quad]0, l_0[\times]0, +\infty[\tag{4.8}$$

$$\rho_2 \tilde{v}_{tt} - \alpha_2 \tilde{v}_{xx} = 0 \quad \text{in} \quad]l_0, l[\times]0, +\infty[\tag{4.9}$$

$$\rho_3 \tilde{w}_{tt} + \gamma_2 \tilde{w} + \alpha_2 \tilde{v}_x(l) = 0 \quad \text{in} \quad]0, +\infty[\tag{4.10}$$

verifying the same boundary and transmission conditions and with the same initial data, where α_1, α_2, ρ_1, ρ_2, ρ_3 and γ_2 are as before. That is, with boundary conditions

$$\tilde{u}(0) = 0, \quad \tilde{v}(l) = \tilde{w} \quad \text{in} \quad]0, +\infty[\tag{4.11}$$

and transmission conditions

$$\tilde{u}(l_0) = \tilde{v}(l_0), \quad \alpha_1 \tilde{u}_x(l_0) = \alpha_2 \tilde{v}_x(l_0) \quad \text{in} \quad]0, +\infty[\tag{4.12}$$

and initial data

$$(\tilde{u}(0), \tilde{v}(0), \tilde{u}_t(0), \tilde{v}_t(0), \tilde{w}(0), \tilde{w}_t(0)) = (u_0, v_0, u_1, v_1, w_0, w_1) \in \breve{\mathscr{H}}. \tag{4.13}$$

The total energy associed with this system is

$$\tilde{E}(t) = \frac{1}{2}\left[\int_0^{l_0}\left[\rho_1|\tilde{u}_t|^2+\alpha_1|\tilde{u}_x|^2\right]dx+\int_{l_0}^{l}\left[\rho_2|\tilde{v}_t|^2+\alpha_2|\tilde{v}_x|^2\right]dx+\rho_3|\tilde{w}_t|^2+\gamma_2|\tilde{w}|^2\right]$$

and it is not difficult to see that

$$\frac{d}{dt}\tilde{E}(t) = 0.$$

Therefore the system is conservative and there is no decay. Now we are in conditions to show the main result of this section.

Theorem 4.4. *The semigroup $T(t)$ associated to system (4.1)-(4.6) is not exponentially stable.*

Proof. The main ideia is to prove that $T(t)$ have the same essential spectral radius of the semigroup associated to conservative system (4.8)-(4.13), that we denote as $T_0(t)$. Here, we use the Weyl's Theorem (Theorem XIII.14, [10]; see also Kato's book [8], Theorem 5.35, p. 244 for details of the proof), which stablish that *if the difference of two operators is compact, then the your essential spectrum radii are equals*. More precisely

Theorem 4.5. *Let S and T two continuous operator over a Banach space X. If $S-T$ is a compact operator, then S and T have the same essential spectrum radius.*

So, we will show that the difference $T(t)-T_0(t)$ is a compact operator; from which we obtain

$$\omega_{ess}(T) = \omega_{ess}(T_0).$$

But since $T_0(t)$ is unitary, then $\omega_{ess}(T_0) = 0$. Denoting by $\omega(T)$ and $\omega_\sigma(\mathscr{B})$ the type of semigroup $T(t)$ and the spectral upper bound of spectrum $\sigma(\mathscr{B})$ respectively, we have that (see [5], Corollary 2.11, p. 258):

$$\omega(T) = \max\{\omega_\sigma(\mathscr{B}), \omega_{ess}(T)\} = 0.$$

This imply that $T(t)$ is not exponentially stable. In fact. Let (u,v,w) and $(\tilde{u},\tilde{v},\tilde{w})$ be the solutions of the systems (4.1)-(4.6) and (4.8)-(4.13), respectively. Denoting by

$$U := u-\tilde{u}, \qquad V := v-\tilde{v}, \qquad W := w-\tilde{w},$$

we have that (U,V,W) is solution of the system

$$\rho_1 U_{tt} - \alpha_1 U_{xx} = 0 \quad \text{in} \quad]0,l_0[\times]0,+\infty[\tag{4.14}$$

$$\rho_2 V_{tt} - \alpha_2 V_{xx} = 0 \quad \text{in} \quad]l_0,l[\times]0,+\infty[\tag{4.15}$$

$$\rho_3 W_{tt} + \gamma_1 w_t + \gamma_2 W + \alpha_2 V_x(l) = 0 \quad \text{in} \quad]0,+\infty[\tag{4.16}$$

with boundary conditions

$$U(0) = 0, \quad V(l) = W \quad \text{in} \quad]0,+\infty[, \tag{4.17}$$

and transmission conditions

$$U(l_0) = V(l_0), \quad \alpha_1 U_x(l_0) = \alpha_2 V_x(l_0) \quad \text{in} \quad]0,+\infty[, \tag{4.18}$$

and initial data

$$(U(0),V(0),U_t(0),V_t(0),W(0),W_t(0)) = (0,0,0,0,0,0) \in \breve{\mathscr{H}}. \tag{4.19}$$

The energy associated with the system (4.14)-(4.19) is given by

$$\mathbb{E}(t) = \frac{1}{2}\left[\int_0^{l_0}\left[\rho_1|U_t|^2+\alpha_1|U_x|^2\right]dx+\int_{l_0}^{l}\left[\rho_2|V_t|^2+\alpha_2|V_x|^2\right]dx+\rho_3|W_t|^2+\gamma_2|W|^2\right].$$

It is easy to verify that

$$\frac{d}{dt}\mathbb{E}(t)+\gamma_1|W_t|^2 = -\gamma_1\tilde{w}_t\overline{W}_t,$$

from where follows

$$\mathbb{E}(t)+\gamma_1\int_0^t|W_t|^2ds = -\gamma_1\int_0^t\tilde{w}_t\overline{W}_t ds. \tag{4.20}$$

Now, let us denote by $\mathscr{U}_{0,n} := (u_{0,n},v_{0,n},u_{1,n},v_{1,n},w_{0,n},w_{1,n})^T$ a bounded sequence of initial data in the phase space $\breve{\mathscr{H}}$. We will show that the corresponding sequence of solutions $\mathscr{U}_n := (U_n,V_n,U_{n,t},V_{n,t},W_n,W_{n,t})^T$ has a subsequence that converges strongly in $\breve{\mathscr{H}}$.

To show this, note that $T(t)\mathscr{U}_{0,n}$ and $T_0(t)\mathscr{U}_{0,n}$ are bounded in $\breve{\mathscr{H}}$. This implies that $\tilde{v}_{n,x}(l)$ is bounded in $L^2(0,T)$, for all $T>0$. Therefore (4.16) implies that $W_{n,t}$ is bounded in $H^1(0,T)$. Since $H^1(0,T)$ has compact embedding in $L^2(0,T)$, it follows that there exist subsequences, we still denote as $(W_n)_n$ and $(\tilde{w}_n)_n$ such that

$$W_{n,t}\longrightarrow W_t \quad \text{strongly in} \quad L^2(0,T),$$

and similarly

$$\tilde{w}_{n,t}\longrightarrow \tilde{w}_t \quad \text{strongly in} \quad L^2(0,T).$$

From the above convergences we have

$$\int_0^T\tilde{w}_{n,t}\overline{W}_{n,t}dt\longrightarrow\int_0^T\tilde{w}_t\overline{W}_t dt. \tag{4.21}$$

Using this convergence in (4.20) it follows that $\|[T(t)-T_0(t)]\mathscr{U}_{0,n}\|_{\breve{\mathscr{H}}}$ converges, which implies that $[T(t)-T_0(t)]\mathscr{U}_{0,n}$ converges strongly in $\breve{\mathscr{H}}$. This means that $T(t)-T_0(t)$ is a compact operator in $\breve{\mathscr{H}}$, and therefore the proof is complete. □

5 POLYNOMIAL DECAY

In this section we show that the solutions of the system (4.1)-(4.6) decays polynomially to zero as $t^{-1/2}$. To show this, we use the Borichev and Tomilov's Theorem (see [2]):

Theorem 5.6. *Let $S(t)$ be a bounded C_0-semigroup on a Hilbert space $\mathscr{H}$ with generator $\mathscr{A}$ such that $i\mathbb{R}\subset\rho(\mathscr{A})$. Then*

$$\frac{1}{|\lambda|^\beta}\|(i\lambda I-\mathscr{A})^{-1}\|_{\mathscr{L}(\mathscr{H})}\le C, \quad \forall\,\lambda\in\mathbb{R} \quad \Longleftrightarrow \quad \|S(t)\mathscr{A}^{-1}\|_{D(\mathscr{A})}\le\frac{C}{t^{1/\beta}}, \quad \forall\, t>0.$$

Our starting point is to study the solution of the resolvent equation

$$i\lambda\mathscr{U} - \mathscr{B}\mathscr{U} = F$$

which is rewritten, in terms of the components, in

$$\begin{aligned}
i\lambda u - U &= f^1 && (5.1)\\
i\lambda v - V &= f^2 && (5.2)\\
i\lambda U - \frac{\alpha_1}{\rho_1}u_{xx} &= f^3 && (5.3)\\
i\lambda V - \frac{\alpha_2}{\rho_2}v_{xx} &= f^4 && (5.4)\\
i\lambda w - W &= f^5 && (5.5)\\
i\lambda W + \frac{\gamma_1}{\rho_3}W + \frac{\gamma_2}{\rho_3}w + \frac{\alpha_2}{\rho_3}v_x(l) &= f^6. && (5.6)
\end{aligned}$$

Taking the inner product with $\mathscr{U}$ and using (4.7) it follows that

$$\gamma_1|W|^2 \le C\|\mathscr{U}\|\|F\|. \tag{5.7}$$

Lemma 5.4. *For $|\lambda|$ large enough, there exist $C>0$ such that*

$$\int_{l_0}^{l}[\rho_2|V|^2+\alpha_2|v_x|^2]dx \le C|\lambda|^2\|\mathscr{U}\|\|F\|+C\|F\|^2.$$

Proof. Multiplying (5.4) by $(x-l_0)\bar{v}_x$ and using (5.2), we have

$$\frac{1}{2}\int_{l_0}^{l}[\rho_2|V|^2+\alpha_2|v_x|^2]dx = \frac{(l-l_0)}{2}\left[\rho_2|W|^2+\alpha_2|v_x(l)|^2\right]+\rho_2\int_{l_0}^{l}(x-l_0)\left[f^4\bar{v}_x+V\bar{f_x^2}\right]dx. \tag{5.8}$$

On the other hand, using (5.6) we get

$$\alpha_2|v_x(l)|^2 \le C|\lambda|^2\|\mathscr{U}\|\|F\|+C\|F\|^2. \tag{5.9}$$

Finally, from (5.7)-(5.9) our conclusion follows. □

Lemma 5.5. *For $|\lambda|$ large enough, there exist $C>0$ such that*

$$\int_{0}^{l_0}[\rho_1|U|^2+\alpha_1|u_x|^2]dx \le C|\lambda|^2\|\mathscr{U}\|\|F\|+C\|F\|^2.$$

Proof. Multiplying equation (5.3) by $x\bar{u}_x$, using (5.1) and the transmission conditions, we get

$$\frac{1}{2}\int_{0}^{l_0}[\rho_1|U|^2+\alpha_1|u_x|^2]dx = \frac{l_0}{2}\left[\rho_1|V(l_0)|^2+\frac{\alpha_2^2}{\alpha_1}|v_x(l_0)|^2\right]+\rho_1\int_{0}^{l_0}x\left[f^3\bar{u}_x+U\bar{f_x^1}\right]dx. \tag{5.10}$$

On the other hand, multiplying equation (5.4) by $(l-x)\bar{v}_x$ and using (5.2) we get

$$\begin{aligned}
\frac{(l-l_0)}{2}\left[\rho_2|V(l_0)|^2+\alpha_2|v_x(l_0)|^2\right] &= \frac{1}{2}\int_{l_0}^{l}[\rho_2|V|^2+\alpha_2|v_x|^2]dx+\\
&+\rho_2\int_{l_0}^{l}(l-x)\left[f^4\bar{v}_x+V\bar{f_x^2}\right]dx.
\end{aligned}$$

From this, and from Lemma 5.4 there exist $C > 0$, such that

$$\rho_2|V(l_0)|^2 + \alpha_2|v_x(l_0)|^2 \leq C|\lambda|^2\|\mathscr{U}\|\|F\| + C\|F\|^2. \tag{5.11}$$

Combining (5.10) and (5.11), our conclusion follows. □

Now we are able to stablish the main result of this section.

Theorem 5.7. *The semigroup $T(t)$ associated to the transmission problem with load tip (4.1)-(4.6) decays polynomially as $t^{-1/2}$. Moreover, if $\mathscr{U}_0 \in D(\mathscr{B}^k)$, then*

$$\|T(t)\mathscr{U}_0\|_{\breve{\mathscr{H}}} \leq \frac{C_k}{t^{k/2}}\|\mathscr{U}_0\|_{D(B^k)}.$$

Proof. The proof of $i\mathbb{R} \subset \rho(\mathscr{B})$ is analogous to the proof of Lemma 3.1. And, from Lemmas 5.4 and 5.5, follows that, for $|\lambda|$ large enough, we have

$$\|\mathscr{U}\|^2_{\breve{\mathscr{H}}} \leq C|\lambda|^2\|\mathscr{U}\|_{\breve{\mathscr{H}}}\|F\|_{\breve{\mathscr{H}}} + C\|F\|^2_{\breve{\mathscr{H}}},$$

and this implies that

$$\|\mathscr{U}\|^2_{\breve{\mathscr{H}}} \leq C|\lambda|^4\|F\|^2_{\breve{\mathscr{H}}},$$

which gives us, for $|\lambda|$ large enough

$$\|(i\lambda I - \mathscr{B})^{-1}F\|_{\breve{\mathscr{H}}} = \|\mathscr{U}\|_{\breve{\mathscr{H}}} \leq C|\lambda|^2\|F\|_{\breve{\mathscr{H}}}.$$

Therefore, from Theorem 5.6 our conclusion follows. □

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