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Quasi-interpolation in Spline Spaces: Local Stability and Approximation Properties

M. E. CASTILLO[1] and E. M. GARAU[2*]

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ABSTRACT. In this work we analyze the approximation error in Sobolev norms for quasi-interpolation operators in spline spaces. We establish in a general way the hypotheses on a quasi-interpolant to achieve the optimal order of approximation. Finally, we propose simple but general constructions of such operators that satisfy the established hypotheses and illustrate their performance through some numerical tests.

Keywords: spline approximation, quasi-interpolation, stability, B-splines.

1 INTRODUCTION

Quasi-interpolation operators represent a practical and efficient method when calculating approximations using spline functions since they present a great simplicity and flexibility in their construction. To define them, B-splines bases are used and the coefficients are chosen locally. That is, each coefficient depends only on the data found in the support of the corresponding B-spline. This localization implies that changes in the data or space of splines have a low computational cost to recalculate the approximation, because the changes are only local.

Quasi-interpolation operators play a fundamental role from several points of view. On the one hand, they constitute a key tool in the theoretical analysis of the approximation power of spline spaces. On the other hand, they provide simple recipes for building good approximations of functions in spline spaces, that are needed in practical algorithms for solving partial differential equations; for example, for imposing boundary conditions and for keeping important information after coarsening in time dependent equations.

2 SPLINE SPACES AND B-SPLINE BASES

Let $[a,b] \subset \mathbb{R}$ and let $Z := \{\zeta_1, \zeta_2, \ldots, \zeta_N\}$ where $\zeta_1 = a$, $\zeta_N = b$ and $\zeta_j < \zeta_{j+1}$, for $j = 1, 2, \ldots, N-1$. Let p be a polynomial degree. We associate a number m_j, called multiplicity,

*Corresponding author: Eduardo M. Garau – E-mail: eduardogarau@gmail.com

[1]Departamento de Matemática, FACET, Universidad Nacional de Tucumán, Tucumán, Argentina – E-mail: mariaemiliacastillo@gmail.com https://orcid.org/0000-0003-1702-4271

[2]Universidad Nacional del Litoral, Consejo Nacional de Investigaciones Científicas y Técnicas, FIQ, Santa Fe, Argentina – E-mail: eduardogarau@gmail.com https://orcid.org/0000-0001-7541-2390

to each breakpoint ζ_j, such that $m_1 = m_N = p+1$ and $1 \leq m_j \leq p+1$, for $j = 2, \ldots, N-1$. Let $\mathscr{S}$ be the space of piecewise polynomials of degree $\leq p$ on Z such that they have $p - m_j$ continuous derivatives at the breakpoint ζ_j. It is known that $\mathscr{S}$ is a vector space of finite dimension with $n := \dim \mathscr{S} = \sum_{i=1}^{N-1} m_i$.

Let $\Xi := \{\xi_j\}_{j=1}^{n+p+1}$ be the associated $(p+1)$-open knot vector, i.e.,

$$\Xi = \{\underbrace{\zeta_1, \ldots, \zeta_1}_{m_1 \text{ times}}, \underbrace{\zeta_2, \ldots, \zeta_2}_{m_2 \text{ times}}, \ldots \underbrace{\zeta_N, \ldots, \zeta_N}_{m_N \text{ times}}\}.$$

Let $\mathscr{B} := \{\beta_1, \beta_2, \ldots, \beta_n\}$ be the B-spline basis of degree p associated to the knot vector Ξ, see e.g., [2, 5]. We remark that B-splines are non-negative, locally supported, and form a convex partition of unity, namely,

- $\beta_i \geq 0$, for $i = 1, 2, \ldots, n$.
- $\operatorname{supp} \beta_i = [\xi_i, \xi_{i+p+1}]$, for $i = 1, 2, \ldots, n$.
- $\displaystyle\sum_{i=1}^{n} \beta_i(x) = 1, \quad \forall x \in [a, b]$.

In Figure 1 we show some examples of cubic B-splines.

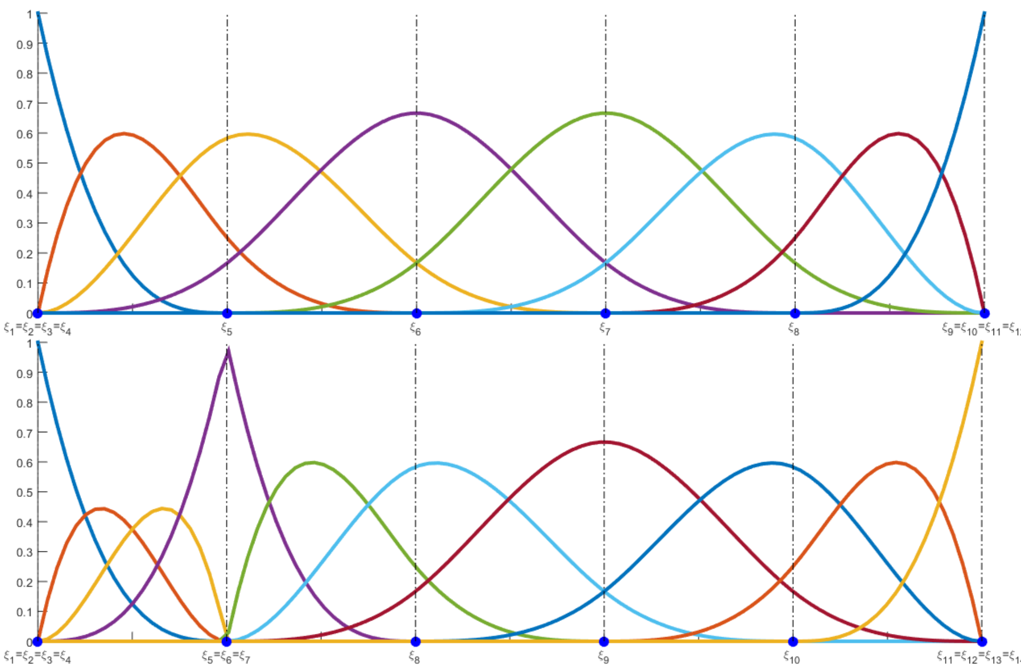

Figure 1: Examples of cubic B-splines of maximum smoothness (top) and with a triple internal knot (bottom).

Let $\mathscr{I}$ be the mesh defined by $\mathscr{I} := \{[\zeta_j, \zeta_{j+1}] \,|\, j = 1, \ldots, N-1\}$. Notice that for each $I = [\zeta_j, \zeta_{j+1}] \in \mathscr{I}$ there exists a unique $k = \sum_{i=1}^{j} m_j$ such that $I = [\xi_k, \xi_{k+1}]$ and $\xi_k \neq \xi_{k+1}$. The union of the supports of the B-splines acting on I identifies the *support extension* $\tilde{I}$, namely $\tilde{I} := [\xi_{k-p}, \xi_{k+p+1}]$.

3 QUASI-INTERPOLATION OPERATORS: STABILITY AND POWER OF APPROXIMATION

Let $E \subset \mathbb{R}$ be a closed interval, $r \in \mathbb{N}_0$ and $1 \leq q \leq \infty$. The Sobolev space $W^{r,q}(E)$ is defined by

$$W^{r,q}(E) := \{u : E \to \mathbb{R} \mid u, Du, \ldots, D^{r-1}u \text{ are absolutely continuous on } E \text{ and } D^r u \in L^q(E)\}.$$

Notice that $C^r(E) \subset W^{r,\infty}(E) \subset W^{r,q}(E) \subset W^{r,1}(E) \subset C^{r-1}(E)$.

A quasi-interpolation operator $\mathscr{Q} : W^{r,q}([a,b]) \to \mathscr{S}$ is defined by

$$\mathscr{Q}f := \sum_{\beta \in \mathscr{B}} \lambda_\beta(f)\beta, \tag{3.1}$$

for some linear functionals $\lambda_\beta : W^{r,q}([a,b]) \to \mathbb{R}$, for $\beta \in \mathscr{B}$.

We consider the following usual assumption which is not too strong.

Assumption 1. *There exists a constant $C_{\mathscr{Q}} > 0$ such that*

$$|\lambda_\beta(f)| \leq C_{\mathscr{Q}} |\operatorname{supp} \beta|^{-\frac{1}{q}} \|f\|_{L^q(\operatorname{supp} \beta)},$$

for all $\beta \in \mathscr{B}$.

3.1 Local stability and approximation properties in L^q-norms

The proofs in this section follow exactly the same lines in [4, Theorem 16], but we include them here for the sake of completeness.

In the following result we prove that Assumption 1 guarantees that the quasi-interpolation operator $\mathscr{Q}$ is locally L^q-stable, $1 \leq q \leq \infty$.

Theorem 3.1 (Local L^q-stability). *Let $\mathscr{Q}$ be a quasi-interpolation operator given by* (3.1) *that satisfies Assumption 1 with constant $C_{\mathscr{Q}}$, for some $1 \leq q \leq \infty$. Then, for $I \in \mathscr{I}$,*

$$\|\mathscr{Q}f\|_{L^q(I)} \leq C_{\mathscr{Q}} \|f\|_{L^q(\tilde{I})}, \qquad \forall f \in L^q(\tilde{I}). \tag{3.2}$$

Proof. Let $I = [\xi_k, \xi_{k+1}] \in \mathscr{I}$ and $x \in I$, then

$$|\mathscr{Q}f(x)| = \left| \sum_{i=k-p}^{k} \lambda_i(f)\beta_i(x) \right| \leq \max_{k-p \leq i \leq k} |\lambda_i(f)| \underbrace{\sum_{i=k-p}^{k} \beta_i(x)}_{=1} \leq C_{\mathscr{Q}} |I|^{-\frac{1}{q}} \|f\|_{L^q(\tilde{I})}.$$

Finally, taking the L^q-norm on I we have that (3.2) holds. □

Let $\ell \in \mathbb{N}_0$ and $\mathscr{P}_\ell$ be the space of polynomials of degree at most ℓ.

Theorem 3.2 (Local approximation in L^q-norms). *Let $\mathscr{Q}$ be a quasi-interpolation operator given by* (3.1) *that satisfies Assumption 1 with constant $C_{\mathscr{Q}}$, for some $1 \le q \le \infty$. Let $0 \le \ell \le p$. If $\mathscr{Q}g = g$, for $g \in \mathscr{P}_\ell$, then,*

$$\|f - \mathscr{Q}f\|_{L^q(I)} \le \frac{(1+C_{\mathscr{Q}})}{\ell!}|\tilde{I}|^{\ell+1}\|D^{\ell+1}f\|_{L^q(\tilde{I})}, \qquad \forall f \in W^{\ell+1,q}(\tilde{I}), \tag{3.3}$$

for all $I \in \mathscr{I}$.

Proof. Let $g \in \mathscr{P}_\ell$. Then, using that $\mathscr{Q}g = g$ and the local L^q-stability of $\mathscr{Q}$ (Theorem 3.1) we have that

$$\|f - \mathscr{Q}f\|_{L^q(I)} \le \|f - g\|_{L^q(I)} + \|\mathscr{Q}(g - f)\|_{L^q(I)} \le (1 + C_{\mathscr{Q}})\|f - g\|_{L^q(\tilde{I})}.$$

If g is the Taylor polynomial of degree ℓ at ξ_{k-p} to f in $\tilde{I}$, the Taylor interpolation error [4, Theorem 15] implies that (3.3) holds. ☐

Theorem 3.3 (Global approximation in L^q). *Let $\mathscr{Q}$ be a quasi-interpolation operator given by* (3.1) *that satisfies Assumption 1 with constant $C_{\mathscr{Q}}$, for some $1 \le q \le \infty$. Let $0 \le \ell \le p$. If $\mathscr{Q}g = g$, for $g \in \mathscr{P}_\ell$, then,*

$$\|f - \mathscr{Q}f\|_{L^q[a,b]} \le C|\mathscr{I}|^{\ell+1}\|D^{\ell+1}f\|_{L^q[a,b]}, \qquad \forall f \in W^{\ell+1,q}([a,b]),$$

where $|\mathscr{I}| := \max\limits_{2 \le i \le N}(\zeta_i - \zeta_{i-1})$ and $C := (2p+1)^{\ell+1+\frac{1}{q}}\frac{(1+C_{\mathscr{Q}})}{\ell!}$.

Proof. The proof follows from Theorem 3.2 using that $|\tilde{I}| \le (2p+1)|\mathscr{I}|$ and the fact that each knot interval I belongs (at most) to $2p+1$ support extensions. ☐

3.2 Local stability and power of approximation in high order norms

Now we extend the results from the previous section by considering Sobolev seminorms in the right hand side of the inequalities.

Following the same lines in the proof of [4, Lemma 4] we obtain the next auxiliary result.

Lemma 3.1. *Let $\mathscr{Q}$ be a quasi-interpolation operator given by* (3.1) *that satisfies Assumption 1 with constant $C_{\mathscr{Q}}$, for some $1 \le q \le \infty$. Then, for $0 \le r \le p$,*

$$\|D^r(\mathscr{Q}f)\|_{L^q(I)} \le C_{p,r}(p+1)C_{\mathscr{Q}}|I|^{-r}\|f\|_{L^q(\tilde{I})}, \qquad \forall f \in L^q(\tilde{I}),$$

for all $I \in \mathscr{I}$, where $C_{p,r} := 2^r\frac{p!}{(p-r)!}$.

Proof. Let $I = [\xi_k, \xi_{k+1}] \in \mathscr{I}$ and $x \in I$, then using the upper bound for the r-th derivative of the B-splines from [4, Proposition 2] we have that

$$|D^r(\mathscr{Q}f)(x)| = \left| \sum_{i=k-p}^{k} \lambda_i(f) D^r \beta_i(x) \right| \leq \max_{k-p\leq i\leq k} |D^r \beta_i(x)| \sum_{i=k-p}^{k} |\lambda_i(f)|$$
$$\leq C_{p,r} |I|^{-r} \sum_{i=k-p}^{k} |\lambda_i(f)|.$$

Finally, using Assumption 1 we conclude the proof. □

Definition 3.1 (Locally quasi-uniform mesh). *The mesh $\mathscr{I}$ is locally quasi-uniform with parameter $\theta > 0$ if*

$$\theta^{-1} \leq \frac{\zeta_j - \zeta_{j-1}}{\zeta_{j+1} - \zeta_j} \leq \theta, \qquad \forall\, j = 2, \ldots, N-1.$$

Recall that $\mathscr{P}_\ell$ denotes the space of polynomials of degree at most ℓ.

Theorem 3.4 (Local stability in high order norms). *Let $\mathscr{Q}$ be a quasi-interpolation operator given by* (3.1) *that satisfies Assumption 1 with constant $C_{\mathscr{Q}}$, for some $1 \leq q \leq \infty$. Let $0 \leq \ell \leq p$. If $\mathscr{Q}g = g$, for $g \in \mathscr{P}_\ell$, then, for $0 \leq r \leq \ell$,*

$$\|D^r(\mathscr{Q}f)\|_{L^q(I)} \leq C_S \|D^r f\|_{L^q(\tilde{I})}, \qquad \forall\, f \in W^{r,q}(\tilde{I}), \tag{3.4}$$

for all $I \in \mathscr{I}$, where the constant $C_S > 0$ depends on $C_{\mathscr{Q}}$, p, r and θ.[1]

Proof. Let $1 \leq r \leq \ell$. Let $f \in W^{r,q}(\tilde{I})$ and let g be the Taylor polynomial of degree $r-1$ at ξ_{k-p} to f in $\tilde{I}$. Since $\mathscr{Q}g = g$, using Lemma 3.1 and the Taylor interpolation error [4, Theorem 15] we have that

$$\|D^r(\mathscr{Q}f)\|_{L^q(I)} \leq \|D^r \mathscr{Q}(f-g)\|_{L^q(I)} \leq \frac{C_{p,r}(p+1)C_{\mathscr{Q}}}{|I|^r} \|f-g\|_{L^q(\tilde{I})}$$
$$\leq \frac{C_{p,r}(p+1)C_{\mathscr{Q}}}{(r-1)!} \frac{|\tilde{I}|^r}{|I|^r} \|D^r f\|_{L^q(\tilde{I})}.$$

Considering that $\frac{|\tilde{I}|}{|I|}$ is bounded above by a constant that depends on p and θ, we conclude that (3.4) holds. □

The next result generalizes [4, Proposition 6] for general quasi-interpolation operators with essentially the same proof.

Theorem 3.5 (Local approximation in high order norms). *Let $\mathscr{Q}$ be a quasi-interpolation operator given by* (3.1) *that satisfies Assumption 1 with constant $C_{\mathscr{Q}}$, for some $1 \leq q \leq \infty$. Let $0 \leq \ell \leq p$. If $\mathscr{Q}g = g$, for $g \in \mathscr{P}_\ell$, then, for $0 \leq r \leq \ell$,*

[1] Notice that (3.4) holds with $C_S = C_{\mathscr{Q}}$ when $r = 0$, due to Theorem 3.1.

$$\|D^r(f-\mathscr{Q}f)\|_{L^q(I)} \le \frac{(1+C_S)}{(\ell-r)!}|\tilde{I}|^{\ell+1-r}\|D^{\ell+1}f\|_{L^q(\tilde{I})}, \qquad \forall f \in W^{\ell+1,q}(\tilde{I}), \tag{3.5}$$

for all $I \in \mathscr{I}$.

Proof. Let $g \in \mathscr{P}_\ell$. Then, using that $\mathscr{Q}g = g$ and the local stability of $\mathscr{Q}$ given in Theorem 3.4 we have that

$$\begin{aligned}\|D^r(f-\mathscr{Q}f)\|_{L^q(I)} &\le \|D^r(f-g)\|_{L^q(I)} + \|D^r\mathscr{Q}(g-f)\|_{L^q(I)}\\ &\le (1+C_S)\|D^r(f-g)\|_{L^q(\tilde{I})}.\end{aligned}$$

If g is the Taylor polynomial of degree ℓ at ξ_{k-p} to f in $\tilde{I}$, the Taylor interpolation error [4, Theorem 15] implies that (3.5) holds. □

4 CONSTRUCTION OF LOCALLY STABLE QUASI-INTERPOLATION OPERATORS

In this section we consider the construction of quasi-interpolation operators based on [1, 3] and apply the results stated in the previous section to analize their local approximation properties in Sobolev norms.

In order to define a quasi-interpolation operator $\mathscr{Q}$ given by

$$\mathscr{Q}f := \sum_{\beta\in\mathscr{B}} \lambda_\beta(f)\beta, \tag{4.1}$$

we need to choose appropriate linear functionals λ_β, for each $\beta \in \mathscr{B}$.

Local approximation method. For each knot interval $I \in \mathscr{I}$, let Π_I be the L^2-projection operator onto $\mathscr{P}_p(I)$, the set of polynomials of degree $\le p$ on I. If $\mathscr{B}_I := \{\beta_1^I, \dots, \beta_{p+1}^I\}$ denotes the set of B-splines restricted to I, we have that $\mathscr{B}_I$ is a basis for $\mathscr{P}_p(I)$ and

$$\Pi_I f = \sum_{i=1}^{p+1} \lambda_i^I(f)\beta_i^I, \qquad \forall f \in L^1(I),$$

for some linear functionals $\lambda_i^I : L^1(I) \to \mathbb{R}$, for $i = 1, \dots, p+1$. Notice that $\Pi_I(f) = f$, for all $f \in \mathscr{P}_p(I)$ which is equivalent to say that the set $\{\lambda_i^I\}_{i=1}^{p+1}$ is a dual basis for $\mathscr{B}_I$ in the sense that $\lambda_i^I(\beta_j^I) = \delta_{ij}$, for $i, j = 1, \dots, p+1$. In view of [1, Theorem 1], there exists a constant $C = C(p,\theta) > 0$ independent of $I \in \mathscr{I}$ such that

$$\max_{i=1,\dots,p+1} |\lambda_i^I(f)| \le C|I|^{-\frac{1}{q}}\|f\|_{L^q(I)}, \qquad f \in L^q(I), \tag{4.2}$$

for q with $1 \le q \le \infty$.

Dual basis for B-splines and quasi-intepolation operator. As explained in [1, Section 6], we can define each λ_β as a convex combination of local projections onto some $I \in \mathscr{I}$ such that $I \subset \operatorname{supp}\beta$. More specifically, for $\beta \in \mathscr{B}$, we define $\mathscr{I}_\beta := \{I \in \mathscr{I} \,|\, I \subset \operatorname{supp}\beta\}$, and for each $I \in \mathscr{I}_\beta$, let $\lambda_\beta^I := \lambda_{i_0}^I$, where $i_0 = i_0(\beta, I)$ with $1 \le i_0 \le p+1$ is such that $\beta_{i_0}^I \equiv \beta$ on I. Now, the functional λ_β is given by

$$\lambda_\beta := \sum_{I \in \mathscr{I}_\beta} c_{I,\beta} \lambda_\beta^I, \tag{4.3}$$

where $c_{I,\beta} \ge 0$, for all $I \in \mathscr{I}_\beta$, and $\sum_{I \in \mathscr{I}_\beta} c_{I,\beta} = 1$.

Then, the quasi-interpolation operator $\mathscr{Q}$ is given by (4.1). Since Π_I is a projection onto $\mathscr{P}_p(I)$, it is easy to check that for any choice of the coefficients $c_{I,\beta}$ we have that $\lambda_{\beta_i}(\beta_j) = \delta_{ij}$, for $\beta_i, \beta_j \in \mathscr{B}$, which in turn implies that $\mathscr{Q}$ is a projection onto the spline space, i.e., $\mathscr{Q}f = f$, whenever $f \in \mathscr{S}$. Moreover, (4.2) and (4.3) imply that Assumption 1 holds with a constant $C_\mathscr{Q} > 0$ that depends on p and θ. Finally, applying Theorems 3.4 and 3.5 we have that $\mathscr{Q}$ satisfies (3.4) and (3.5), which generalize [1, Theorem 2] when considering high order norms in the left hand side.

5 NUMERICAL TESTS

In this section we propose two specific ways of defining the coefficients $c_{I,\beta}$, for $I \in \mathscr{I}_\beta$, in the framework of the previous section. Each of them, in turn, gives rise a particular quasi-interpolation operator.

(i) Given $\beta \in \mathscr{B}$, we let the coefficient associated to the central knot interval in the support of β be equal 1, whenever the mumber of knot intervals within its support is odd; whereas we define as $\frac{1}{2}$ the coefficients associated to the two central knot intervals if such number is even (cf. Figure 2). More precisely, let us assume that $\mathscr{I}_\beta$ consists of $k = k(\beta)$ consecutive knot intervals, namely, $I_1, I_2, \ldots, I_k$. Then, we let $c_{I_{\frac{k+1}{2}},\beta} = 1$ if k is odd, and $c_{I_{\frac{k}{2}},\beta} = c_{I_{\frac{k}{2}+1},\beta} = \frac{1}{2}$ if k is even.

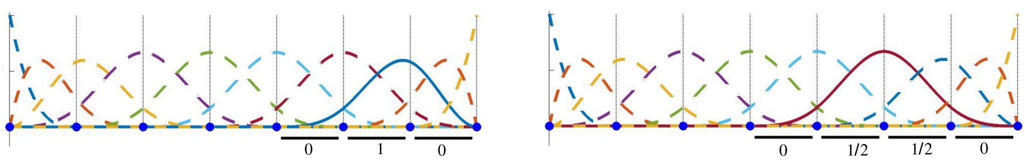

Figure 2: The procedure from (i) for the choice of coefficients $c_{I,\beta}$ in (4.3) for some B-splines β (solid lines).

(ii) We define $c_{I,\beta} = \dfrac{\int_I \beta(x)\,dx}{\int_{\operatorname{supp}\beta} \beta(x)\,dx}$, for $I \in \mathscr{I}_\beta$. The operator defined with this specific choice of coefficients $c_{I,\beta}$ has been introduced in [6] where it was called Bézier projection.

In Figure 3 we consider the L^2- error in approximations of the function $f(x) = \arctan(25x)$, for $x \in [-1,1]$, in spline spaces of degree 4 (left) and degree 5 (right) with maximum smoothness defined onto uniform partitions. We consider four kinds of approximations: the L^2-projection Π_{L^2} onto the spline space, the quasi-interpolant $\mathscr{Q}_0$ defined in [4, Section 1.5.3.1], and the operators $\mathscr{Q}_1$ and $\mathscr{Q}_2$ obtained by picking the coefficients $c_{I,\beta}$ as described in (i) and (ii), respectively. In these cases, we obtain the optimal orders of convergence for all the considered methods. However, it is worth to notice that the behaviour of $\mathscr{Q}_1$ and especially $\mathscr{Q}_2$ are quite similar to the best approximation Π_{L^2}.

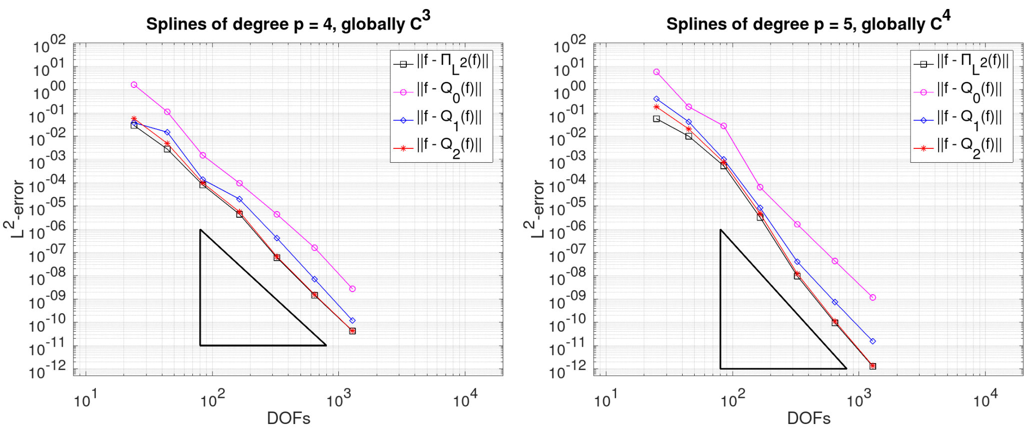

Figure 3: L^2-error in different approximation methods for f in spline spaces of degrees 4 (left) and 5 (right) with maximum smoothness onto uniform meshes.

In Figure 4 we consider the L^2- error associated to the four mentioned operators in the approximation of the function $g(x) = \sin(x^{0.35}e^{0.22x})$, for $x \in [0,10]$, in spline spaces of degree 4 (left) and degree 5 (right) with maximum smoothness defined onto uniform partitions. Now, we obtain a suboptimal order of convergence in all cases due to the lack of regularity of the function g. But we notice that, as in the previous case, the errors for $\mathscr{Q}_1$ and $\mathscr{Q}_2$ are basically like the error for Π_{L^2}.

In Figure 5 we consider the number of degrees of freedom (DOFs) vs. $\|g - \mathscr{Q}_1(g)\|_{L^2}$. We compare the performance for splines of degree 4 (left) and degree 5 (right) with maximum smoothness with globally continuous splines of the same degrees defined both on the same type of mesh. First, we consider uniform meshes and we obtain suboptimal rates of convergence as before, but in this case the space of splines C^0 works better. For example, in order to get an error $\approx 10^{-4}$ using splines of degree 4 with maximum smoothness we requiere 2564 DOFs whereas using splines C^0 we only need 641 DOFs, i.e., four times less. Additionally, in Figure 5 we consider also adaptive meshes. Given the mesh $\mathscr{I}$, we compute the local errors $\|g - \mathscr{Q}_1(g)\|_{L^2(I)}$, for $I \in \mathscr{I}$. Then, we refine dyadically the elements where the local error is greater than $0.3 \max_{I \in \mathscr{I}} \|g - \mathscr{Q}_1(g)\|_{L^2(I)}$. We notice that using adaptive meshes we recover the

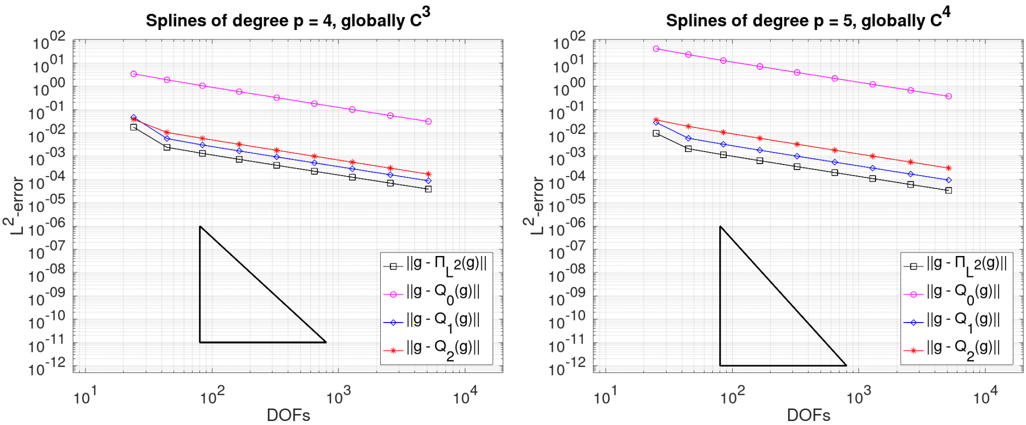

Figure 4: L^2-error in different approximation methods for g in spline spaces of degrees 4 (left) and 5 (right) with maximum smoothness onto uniform meshes.

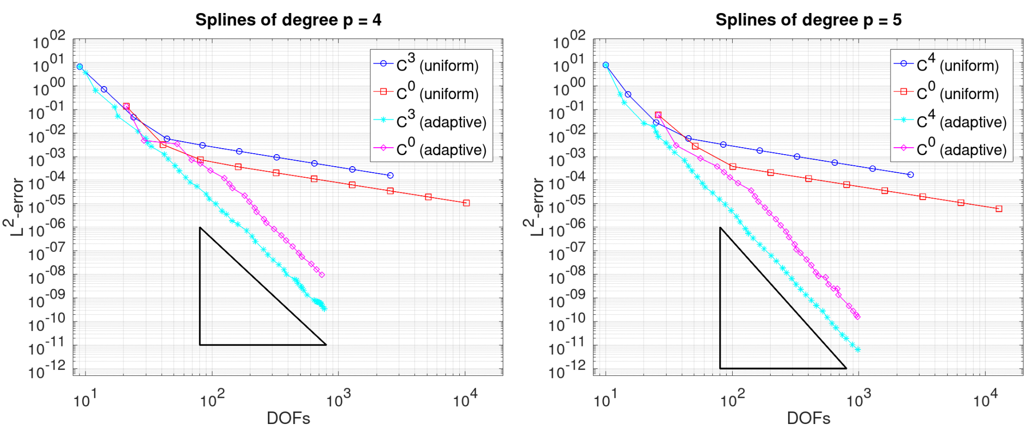

Figure 5: The decay of $\|g - \mathscr{Q}_1(g)\|_{L^2}$ for splines of degrees 4 (left) and 5 (right) with maximum smoothness and globally continuous splines of the same degrees onto uniform and adaptive meshes.

optimal order of convergence in all cases. But now, using splines of maximum smoothness we can achieve a given accuracy with half of the DOFs used by C^0 splines.

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