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# An A Posteriori Error Estimator for a Non Homogeneous Dirichlet Problem Considering a Dual Mixed Formulation

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**ABSTRACT.** In this paper, we describe an a posteriori error analysis for a conforming dual mixed scheme of the Poisson problem with non homogeneous Dirichlet boundary condition. As a result, we obtain an a posteriori error estimator, which is proven to be reliable and locally efficient with respect to the usual norm on $H(\mathrm{div};\Omega)\times L^2(\Omega)$. We remark that the analysis relies on the standard Ritz projection of the error, and take into account a kind of a quasi-Helmholtz decomposition of functions in $H(\mathrm{div};\Omega)$, which we have established in this work. Finally, we present one numerical example that validates the well behavior of our estimator, being able to identify the numerical singularities when they exist.

**Keywords:** mixed finite element methods, a posteriori error estimator, reliability, efficiency.

## 1 INTRODUCTION

It is well known that when the solution of a variational formulation obtained by applying a finite element method, is not smooth enough, the quality of approximation could be not good enough. This motivates us to derive an a posteriori error estimator, which is reliable and efficiency. This would allow us to establish that the estimator behaves as the error of the method, which in general is not known. Then, considering an appropriate adaptive refinement algorithm, we can obtain approximations of the formulation, of better quality, by detecting the region where this estimator is more dominant. In the context of mixed finite element methods, there are a lot of references dedicated to the a posteriori error analysis. For instance, in [1] an a posteriori error estimator only for the flux unknown is derived, using Raviart-Thomas (RT) or Brezzi-Douglas-Marini

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(BDM) as its space of approximation. The analysis that yields this estimator, relies on a classical Helmholtz decomposition. On the other hand, in [10], the authors present two a posteriori error estimators for a dual mixed formulation for the Poisson problem, approximating the flux in the Raviart-Thomas space. In this case, the derivation of the estimator is obtained under a saturation assumption. This requirement is circumvented in [12], where a reliable and efficient a posteriori error estimator for the natural norm, is derived. We remark that four different kind of a posteriori error estimators for Raviart-Thomas mixed finite elements, are provided in [23]. Concerning second order elliptic equation with mixed boundary condition, in [17] the authors developed an a posterior error analysis for the mixed finite element method with a Lagrange multiplier.

In [9], an a posteriori error analysis for an augmented mixed formulation of the Poisson problem with mixed boundary conditions, is developed. This is performed with the help of the Ritz projection of the error, and covers the reliability and efficiency of the estimator. It is important to remark that this technique has been successfully applied to other problems, such as the the Brinkman model in [2], the Darcy flow in [6] and [7], the Stokes system in [3] and [5], and the Oseen equations in [8], for example.

In this paper, we deduce a reliable and efficient residual a posteriori error estimator for the Poisson problem with non homogeneous Dirichlet boundary condition, considering a dual mixed finite element method. To achieve this, we take into account the Ritz projection of the error, measured in the standard $H(\text{div};\Omega)\times L^2(\Omega)$ norm. We also establish another kind of quasi Helmholtz decomposition of $H(\text{div};\Omega)$ in the plane. We remark that in this process, no saturation assumption is required, and its extension to 3D case is not difficult. We remark that in [1] the a posteriori error analysis is performed to a homogeneous Dirichlet problem, focusing in obtain an estimator for the $H(\text{div};\Omega)$ norm of the flux error. Then, the results of the current work can be seen as a natural extension of what is described in [1], since we deduce an a posteriori error estimator for the norm of the error of the flux and potential unknowns, that is reliable and efficient.

The rest of the article is organized as follows: In Section 2 we present the model problem, as well as the corresponding dual mixed formulations, at continuous and discrete levels. Next, the a posteriori error analysis with non homogeneous Dirichlet is described in Sections 3. This includes the introduction of the Ritz projection of the error, as well as the key tool for deducing a reliable a posteriori error estimator: a quasi-Helmholtz decomposition of functions in $H(\text{div};\Omega)$. Finally, one numerical example confirming our theoretical results are reported in Section 4. We end this introduction with some notation to be used throughout the paper. Given any Hilbert space $H$, we denote by $H^2$ the space of vectors of order 2 with entries in $H$. Finally, we use $C$ or $c$, with or without subscripts, to denote generic constants, independent of the discretization parameter, that may take different values at different occurrences.

## 2 MODEL PROBLEM AND VARIATIONAL FORMULATIONS

Let $\Omega$ be a bounded and simply connected domain in $\mathbb{R}^2$ with polygonal boundary $\Gamma$. Then, given $f\in L^2(\Omega)$ and $g\in H^{1/2}(\Gamma)$, we consider the model problem: Find $u\in H^1(\Omega)$ such that $-\Delta u = f$

in $\Omega$ and $u = g$ on $\Gamma$. Since we are interested in dual mixed methods, we rewrite the Dirichlet problem as the first order system: Find $(\boldsymbol{\sigma}, u)$ such that $\boldsymbol{\sigma} = -\nabla u$ in $\Omega$, $\operatorname{div}(\boldsymbol{\sigma}) = f$ in $\Omega$, and $u = g$ on $\Gamma$. Hence, proceeding in the usual way, we arrive to the following dual mixed variational formulation: Find $(\boldsymbol{\sigma}, u) \in H(\operatorname{div};\Omega) \times L^2(\Omega)$ such that

$$\begin{array}{lcll} a(\boldsymbol{\sigma}, \boldsymbol{\tau}) - b(u, \boldsymbol{\tau}) & = & -\langle \boldsymbol{\tau} \cdot \boldsymbol{n}, g \rangle & \forall \boldsymbol{\tau} \in H(\operatorname{div};\Omega)\,, \\ -b(w, \boldsymbol{\sigma}) & = & -\displaystyle\int_\Omega f w & \forall w \in L^2(\Omega)\,, \end{array} \tag{2.1}$$

where $\langle \cdot, \cdot \rangle$ denotes the duality pairing between $H^{-1/2}(\Gamma)$ and $H^{1/2}(\Gamma)$ with respect to $L^2(\Gamma)$-inner product, and the bilinear forms $a : H(\operatorname{div};\Omega) \times H(\operatorname{div};\Omega) \to \mathbb{R}$ and $b : L^2(\Omega) \times H(\operatorname{div};\Omega) \to \mathbb{R}$, are given by $a(\boldsymbol{\zeta}, \boldsymbol{\tau}) := \int_\Omega \boldsymbol{\zeta} \cdot \boldsymbol{\tau}$ and $b(w, \boldsymbol{\tau}) := \int_\Omega w \operatorname{div}(\boldsymbol{\tau})$, respectively. Thanks to the classical Babuška-Brezzi theory (cf. Section 5 in [16]), it can be shown that there exists a unique pair $(\boldsymbol{\sigma}, u) \in H(\operatorname{div};\Omega) \times L^2(\Omega)$ solution of (2.1). For the discretization, we assume that $\Omega$ is a polygonal region and let $\{\mathscr{T}_h\}_{h>0}$ be a regular family of triangulations of $\bar{\Omega}$ such that $\bar{\Omega} = \cup\{\, T \,:\, T \in \mathscr{T}_h \,\}$. For any triangle $T \in \mathscr{T}_h$, we denote by $h_T$ its diameter and define the mesh size $h := \max\{\, h_T \,:\, T \in \mathscr{T}_h \,\}$. In addition, given an integer $\ell \geq 0$ and a subset $S$ of $\mathbb{R}^2$, we denote by $\mathscr{P}_\ell(S)$ the space of polynomials in two variables defined in $S$ of total degree at most $\ell$, and for each $T \in \mathscr{T}_h$, we define the local Raviart-Thomas space of order $\kappa \geq 0$ (cf. [20]), $\mathscr{RT}_\kappa(T) := [\mathscr{P}_\kappa(T)]^2 \oplus \mathscr{P}_\kappa(T)\boldsymbol{x} \subseteq [\mathscr{P}_{\kappa+1}(T)]^2 \quad \forall \boldsymbol{x} \in T$. Then, given an integer $r \geq 0$, we define the finite element subspaces $H^{\boldsymbol{\sigma}}_{h,r} := \{\, \boldsymbol{\tau}_h \in H(\operatorname{div};\Omega) \,:\, \boldsymbol{\tau}_h|_T \in \mathscr{RT}_r(T)\,, \quad \forall T \in \mathscr{T}_h \,\}$ and $H^u_{h,r} := \left\{\, v_h \in L^2(\Omega) \,:\, v_h|_T \in \mathscr{P}_r(T)\,, \quad \forall T \in \mathscr{T}_h \,\right\}$. Under these assumptions, and applying a discrete version of the Babuška-Brezzi theory (see Section 5 in [16]), we can ensure that there exists only one $(\boldsymbol{\sigma}_h, u_h) \in H^{\boldsymbol{\sigma}}_{h,r} \times H^u_{h,r}$ such that

$$\begin{array}{lcll} a(\boldsymbol{\sigma}_h, \boldsymbol{\tau}_h) - b(u_h, \boldsymbol{\tau}_h) & = & -\langle \boldsymbol{\tau}_h \cdot \boldsymbol{n}, g \rangle & \forall \boldsymbol{\tau}_h \in H^{\boldsymbol{\sigma}}_{h,r}\,, \\ -b(w_h, \boldsymbol{\sigma}_h) & = & -\displaystyle\int_\Omega f w_h & \forall w_h \in H^u_{h,r}\,. \end{array} \tag{2.2}$$

Moreover, the following result is established.

**Theorem 2.1.** *Let $(\boldsymbol{\sigma}, u)$ and $(\boldsymbol{\sigma}_h, u_h)$ be the solutions of* (2.1) *and* (2.2)*, respectively. If $(\boldsymbol{\sigma}, u) \in [H^r(\Omega)]^2 \times H^r(\Omega)$, and $\operatorname{div}(\boldsymbol{\sigma}) \in H^r(\Omega)$, $0 < r \leq k+1$, then there exists $C > 0$, independent of the mesh size, such that*

$$||\boldsymbol{\sigma} - \boldsymbol{\sigma}_h||_{H(\operatorname{div};\Omega)} + ||u - u_h||_{L^2(\Omega)} \leq C h^r \left( |\boldsymbol{\sigma}|_{[H^r(\Omega)]^2} + |u|_{H^r(\Omega)} + |\operatorname{div}(\boldsymbol{\sigma})|_{H^r(\Omega)} \right).$$

**Proof.** We refer to the proofs of Theorems 3.2 and 3.3 in [16], as well as the classical error estimates for the $L^2$-orthogonal projection onto $\mathscr{P}_r$. We omit further details. □

## 3 A POSTERIORI ERROR ANALYSIS

In this section, we follow [4] (see also [5]), and develop an a posteriori error analysis for the discrete scheme (2.2), taking into account an appropriate Ritz projection of the error and a quasi-Helmholtz decomposition. We first introduce some notations and results, concerning the Clément and Raviart-Thomas interpolation operators.

### 3.1 Notation and some well known results

Given $T \in \mathscr{T}_h$, we let $E(T)$ be the set of its edges. By $E_h$ we denote the set of all edges (counted once) induced by the triangulation $\mathscr{T}_h$. Then, we write $E_h = E_I \cup E_\Gamma$, where $E_I := \{e \in E_h \,:\, e \subseteq \Omega\}$ and $E_\Gamma := \{e \in E_h \,:\, e \subseteq \Gamma\}$. Similarly, $N_h$ will denote the list of all vertices (counted once) induced by the triangulation $\mathscr{T}_h$. Then we define $N_I := N_h \cap \Omega$ and $N_\Gamma := \{\boldsymbol{x} \in N_h \,:\, \boldsymbol{x} \in \Gamma\}$. As a result, we have that $N_h = N_I \cup N_\Gamma$. In addition, for each $T \in \mathscr{T}_h$, $N(T) := \{\boldsymbol{x} \in N_h \,:\, \boldsymbol{x} \text{ is a vertex of } T\}$, and for each $e \in E_h$, $N(e) := \{\boldsymbol{x} \in N_h \,:\, \boldsymbol{x} \text{ is a vertex of } e\}$. Now, given $\boldsymbol{x} \in N_h$, $T \in \mathscr{T}_h$ and $e \in E_h$, we set

$$\omega(\boldsymbol{x}) := \bigcup_{\substack{T \in \mathscr{T}_h \\ \boldsymbol{x} \in N(T)}} T\,, \quad \omega(e) := \bigcup_{\boldsymbol{x} \in N(e)} \omega(\boldsymbol{x})\,, \quad \omega(T) := \bigcup_{\boldsymbol{x} \in N(T)} \omega(\boldsymbol{x})\,.$$

Also, for each $T \in \mathscr{T}_h$, we fix a unit normal exterior vector $\boldsymbol{n}_T := (n_1, n_2)^{\mathrm{t}}$, and let $\boldsymbol{t}_T := (-n_2, n_1)^{\mathrm{t}}$ be the corresponding fixed unit tangential vector along $\partial T$. From now on, when no confusion arises, we simply write $\boldsymbol{n}$ and $\boldsymbol{t}$ instead of $\boldsymbol{n}_T$ and $\boldsymbol{t}_T$, respectively. In addition, let $q$ and $\boldsymbol{\tau}$ be scalar - and vector -valued functions, respectively, that are smooth inside each element $T \in \mathscr{T}_h$. We denote by $(q_{T,e}, \boldsymbol{\tau}_{T,e})$ the restriction of $(q_T, \boldsymbol{\tau}_T)$ to $e$. Then, given $e \in E_I$, we define the jump of $q$ and of the tangential component of $\boldsymbol{\tau}$ at $\boldsymbol{x} \in e$, by

$$[\![q]\!] := q_{T,e} - q_{T',e}\,, \quad [\![\boldsymbol{\tau} \cdot \boldsymbol{t}]\!] := \boldsymbol{\tau}_{T,e} \cdot \boldsymbol{t}_T + \boldsymbol{\tau}_{T',e} \cdot \boldsymbol{t}_{T'}\,,$$

where $T$ and $T'$ are the two elements in $\mathscr{T}_h$ sharing the edge $e \in E_I$. On boundary edges $e \in E_\Gamma$, we set $[\![\boldsymbol{\tau} \cdot \boldsymbol{t}]\!] := \boldsymbol{\tau}_{T,e} \cdot \boldsymbol{t}_T$, where $T \in \mathscr{T}_h$ is such that $\partial T \cap e \neq \emptyset$. Finally, given a smooth scalar field $v$ and a vector field $\boldsymbol{\tau} = (\tau_1, \tau_2)^{\mathrm{t}}$, we define

$$\mathbf{curl}(v) := \begin{pmatrix} \frac{\partial v}{\partial x_2} \\ -\frac{\partial v}{\partial x_1} \end{pmatrix} \quad \text{and} \quad \mathrm{rot}(\boldsymbol{\tau}) := \frac{\partial \tau_2}{\partial x_1} - \frac{\partial \tau_1}{\partial x_2}\,.$$

Next, we introduce the Clément interpolation operator $I_h : H^1(\Omega) \to X_h$ (cf. [15]), where $X_h := \{v_h \in H^1(\Omega) \,:\, v_h\big|_T \in \mathscr{P}_1(T)\,,\ \forall T \in \mathscr{T}_h\}$. The following lemma establishes the main local approximation properties of $I_h$.

**Lemma 3.1.** *There exist constants $c_1, c_2 > 0$, independent of h, such that for all $v \in H^1(\Omega)$, there holds*

$$\|v - I_h(v)\|_{H^m(T)} \leq c_1\, h_T^{1-m}\, |v|_{H^1(\omega(T))}\,, \quad \forall m \in \{0,1\}\,, \forall T \in \mathscr{T}_h\,,$$

*and*

$$\|v - I_h(v)\|_{L^2(e)} \leq c_2\, h_e^{1/2}\, |v|_{H^1(\omega(e))} \quad \forall e \in E_h\,,$$

*where $h_e$ denotes the length of the side $e \in E_h$.*

**Proof.** We refer to [15]. □

On the other hand, we also need to introduce the Raviart-Thomas interpolation operator (see [11, 20]), $\pi_h^r : [H^1(\Omega)]^2 \to H_h^{\boldsymbol{\sigma}}$, which given $\boldsymbol{\tau} \in [H^1(\Omega)]^2$, $\pi_h^r \boldsymbol{\tau} \in H_h^{\boldsymbol{\sigma}}$ is characterized by the following conditions:

$$\forall e \in E_h : \forall q \in \mathscr{P}_r(e) : \int_e \pi_h^r(\boldsymbol{\tau}) \cdot \boldsymbol{n}\, q = \int_e \boldsymbol{\tau} \cdot \boldsymbol{n}\, q, \quad \text{when } r \geq 0, \tag{3.1}$$

and

$$\forall T \in \mathscr{T}_h : \forall \boldsymbol{\rho} \in [\mathscr{P}_{r-1}(T)]^2 : \int_T \pi_h^r(\boldsymbol{\tau}) \cdot \boldsymbol{\rho} = \int_T \boldsymbol{\tau} \cdot \boldsymbol{\rho}, \quad \text{when } r \geq 1. \tag{3.2}$$

The operator $\pi_h^r$ satisfies the following approximation properties.

**Lemma 3.2.** *There exist constants $c_3, c_4, c_5 > 0$, independent of $h$, such that for all $T \in \mathscr{T}_h$*

$$\forall \boldsymbol{\tau} \in [H^m(\Omega)]^2 : \|\boldsymbol{\tau} - \pi_h^r(\boldsymbol{\tau})\|_{[L^2(T)]^2} \leq c_3\, h_T^m\, |\boldsymbol{\tau}|_{[H^m(T)]^2} \quad 1 \leq m \leq r+1, \tag{3.3}$$

*for all $\boldsymbol{\tau} \in [H^m(\Omega)]^2$ with $\operatorname{div}(\boldsymbol{\tau}) \in H^m(\Omega)$,*

$$||\operatorname{div}(\boldsymbol{\tau} - \pi_h^r(\boldsymbol{\tau}))||_{L^2(T)} \leq c_4\, h_T^m\, |\operatorname{div}(\boldsymbol{\tau})|_{H^m(T)}, \quad 0 \leq m \leq r+1, \tag{3.4}$$

*and for any $\boldsymbol{\tau} \in [H^1(\Omega)]^2$*

$$\forall e \in E_h : \|\boldsymbol{\tau} \cdot \boldsymbol{n} - \pi_h^r(\boldsymbol{\tau}) \cdot \boldsymbol{n}\|_{L^2(e)} \leq c_5\, h_e^{1/2}\, \|\boldsymbol{\tau}\|_{[H^1(T_e)]^2}, \tag{3.5}$$

*where $T_e \in \mathscr{T}_h$, such that it contains $e$ on its boundary.* **Proof.** See e.g. [11] or [20]. □

In addition, the interpolation operator $\pi_h^r$ can also be defined as a bounded linear operator from the larger space $[H^s(\Omega)]^2 \cap H(\operatorname{div};\Omega)$ into $H_h^{\boldsymbol{\sigma}}$, for all $s \in (1/2, 1]$ (see, e.g. Theorem 3.16 in [19]). In this case, there holds the following interpolation error estimate

$$\forall T \in \mathscr{T}_h : \|\boldsymbol{\tau} - \pi_h^r(\boldsymbol{\tau})\|_{[L^2(T)]^2} \leq C\, h_T^s \left\{ \|\boldsymbol{\tau}\|_{[H^s(T)]^2} + \|\operatorname{div}(\boldsymbol{\tau})\|_{L^2(T)} \right\}.$$

Taking into account (3.1) and (3.2), it is not difficult to show that

$$\operatorname{div}(\pi_h^r(\boldsymbol{\tau})) = P_h^r(\operatorname{div}(\boldsymbol{\tau})), \tag{3.6}$$

where $P_h^r : L^2(\Omega) \to H_h^u$ is the $L^2-$orthogonal projector. On the other hand, it is well known (see, e.g. [14]) that for each $v \in H^m(\Omega)$, with $0 \leq m \leq r+1$, there exists $C > 0$, independent of $h$, such that

$$\forall T \in \mathscr{T}_h : \|v - P_h^r(v)\|_{L^2(T)} \leq C\, h_T^m\, |\boldsymbol{v}|_{H^m(T)}. \tag{3.7}$$

### 3.2 Reliability of the estimator

Let $(\boldsymbol{\sigma}, u) \in \boldsymbol{\Sigma} := H(\operatorname{div};\Omega) \times L^2(\Omega)$ and $(\boldsymbol{\sigma}_h, u_h) \in H_{h,r}^{\boldsymbol{\sigma}} \times H_{h,r}^u \subseteq \boldsymbol{\Sigma}$ be the unique solution to problems (2.1) and (2.2), respectively. We provide $\boldsymbol{\Sigma}$ with its usual inner product

$$\langle (\boldsymbol{\rho}, z), (\boldsymbol{\tau}, v) \rangle_{\boldsymbol{\Sigma}} := (\boldsymbol{\rho}, \boldsymbol{\tau})_{H(\operatorname{div};\Omega)} + (z, v)_{L^2(\Omega)} \quad \forall (\boldsymbol{\rho}, z), (\boldsymbol{\tau}, v) \in \boldsymbol{\Sigma},$$

which induces the norm

$$||(\boldsymbol{\tau},v)||_{\boldsymbol{\Sigma}} := \left(||\boldsymbol{\tau}||^2_{H(\mathrm{div};\Omega)} + ||v||^2_{L^2(\Omega)}\right)^{1/2} \quad \forall\,(\boldsymbol{\tau},v) \in \boldsymbol{\Sigma}.$$

Next, we consider the Ritz projection of the error with respect to $\langle\cdot,\cdot\rangle_{\boldsymbol{\Sigma}}$ as the unique element $(\bar{\boldsymbol{\sigma}},\bar{u}) \in \boldsymbol{\Sigma}$, such that

$$\forall\,(\boldsymbol{\tau},v) \in \boldsymbol{\Sigma} : \langle(\bar{\boldsymbol{\sigma}},\bar{u}),(\boldsymbol{\tau},v)\rangle_{\boldsymbol{\Sigma}} = A((\boldsymbol{\sigma}-\boldsymbol{\sigma}_h,u-u_h),(\boldsymbol{\tau},v))\,, \tag{3.8}$$

where the global bilinear form $A : \boldsymbol{\Sigma}\times\boldsymbol{\Sigma} \to \mathbb{R}$ arises from the variational formulation (2.1), after adding its equations, that is

$$A((\boldsymbol{\rho},w),(\boldsymbol{\tau},v)) := a(\boldsymbol{\rho},\boldsymbol{\tau}) - b(w,\boldsymbol{\tau}) - b(v,\boldsymbol{\rho}) \qquad \forall\,(\boldsymbol{\rho},w),(\boldsymbol{\tau},v) \in \boldsymbol{\Sigma}.$$

We remark that the existence and uniqueness of $(\bar{\boldsymbol{\sigma}},\bar{u}) \in \boldsymbol{\Sigma}$ is guaranteed by the Lax-Milgram Lemma. Moreover, we point out that the properties of the bilinear forms $a(\cdot,\cdot)$ and $b(\cdot,\cdot)$ implies that $A(\cdot,\cdot)$ satisfies a global inf-sup condition, i.e., there exist $\alpha > 0$ such that

$$\alpha\,||(\boldsymbol{\zeta},w)||_{\boldsymbol{\Sigma}} \le \sup_{\theta\neq(\boldsymbol{\tau},v)\in\boldsymbol{\Sigma}} \frac{A((\boldsymbol{\zeta},w),(\boldsymbol{\tau},v))}{\|(\boldsymbol{\tau},v)\|_{\boldsymbol{\Sigma}}}\,, \qquad \forall(\boldsymbol{\zeta},w) \in \boldsymbol{\Sigma}.$$

This particularity allows us to bound the error in terms of the solution of its Ritz projection, as follows:

$$\alpha\,||(\boldsymbol{\sigma}-\boldsymbol{\sigma}_h,u-u_h)||_{\boldsymbol{\Sigma}} \le \sup_{\theta\neq(\boldsymbol{\tau},v)\in\boldsymbol{\Sigma}} \frac{A((\boldsymbol{\sigma}-\boldsymbol{\sigma}_h,u-u_h),(\boldsymbol{\tau},v))}{\|(\boldsymbol{\tau},v)\|_{\boldsymbol{\Sigma}}} = ||(\bar{\boldsymbol{\sigma}},\bar{u})||_{\boldsymbol{\Sigma}}\,. \tag{3.9}$$

Then, according to (3.9), and with the purpose of obtaining a reliable a posteriori error estimate for the discrete scheme (2.2), it is enough to bound from above the Ritz projection of the error. To this aim, the next result will be useful, and can be seen as a kind of a *quasi-Helmholtz decomposition* of functions in $H(\mathrm{div};\Omega)$.

**Lemma 3.3.** *For each $\boldsymbol{\tau} \in H(\mathrm{div};\Omega)$, there exist $\chi \in H^1(\Omega)$ and $\boldsymbol{\Phi} \in [H^1_0(\Omega)]^2$, such that*

$$\boldsymbol{\tau} = \mathbf{curl}(\chi) + \boldsymbol{\Phi} + \frac{d}{2}\begin{pmatrix} x_1-a \\ x_2-b \end{pmatrix}, \tag{3.10}$$

*where $(a,b)^{\mathrm{t}}$ is any fixed point belonging to $\Omega$, and $d := \frac{1}{|\Omega|}\int_\Omega \mathrm{div}(\boldsymbol{\tau})$. In addition, there exists $C > 0$, such that*

$$|\chi|_{H^1(\Omega)} + \|\boldsymbol{\Phi}\|_{[H^1(\Omega)]^2} \le C\,\|\boldsymbol{\tau}\|_{H(\mathrm{div};\Omega)}\,. \tag{3.11}$$

**Proof.** We first introduce the space $M := \{\boldsymbol{\zeta} \in H(\mathrm{div};\Omega) : \int_\Omega \mathrm{div}(\boldsymbol{\zeta}) = 0\}$. Next, for each $\boldsymbol{\tau} \in H(\mathrm{div};\Omega)$, we decompose $\mathrm{div}(\boldsymbol{\tau}) = \mathrm{div}(\tilde{\boldsymbol{\tau}}) + d$, where $\tilde{\boldsymbol{\tau}} := \boldsymbol{\tau} - \frac{d}{2}\begin{pmatrix} x_1-a \\ x_2-b \end{pmatrix} \in M$.

We remark that $\|\mathrm{div}(\boldsymbol{\tau})\|_{0,\Omega}^2 = \|\mathrm{div}(\tilde{\boldsymbol{\tau}})\|_{0,\Omega}^2 + d^2\,|\Omega|$. Then, since $\mathrm{div}(\tilde{\boldsymbol{\tau}}) \in L_0^2(\Omega)$, and invoking Corollary I.2.4 in [18], there exists $\boldsymbol{\Phi} \in [H_0^1(\Omega)]^2$ such that $\mathrm{div}(\boldsymbol{\Phi}) = \mathrm{div}(\tilde{\boldsymbol{\tau}})$ in $\Omega$ and $\|\boldsymbol{\Phi}\|_{1,\Omega} \leq c\,\|\mathrm{div}(\boldsymbol{\tau})\|_{0,\Omega}$. This implies that

$$\mathrm{div}\left(\boldsymbol{\tau} - \boldsymbol{\Phi} - \frac{d}{2}(x_1-a,x_2-b)^t\right) = 0 \quad \text{in} \quad \Omega$$
$$\text{and} \quad \left\langle \left(\boldsymbol{\tau} - \boldsymbol{\Phi} - \frac{d}{2}(x_1-a,x_2-b)^t\right)\cdot \boldsymbol{n}, 1\right\rangle_\Gamma = 0\,,$$

where $(a,b)^{\mathrm{t}}$ is a fixed point belonging to $\Omega$. Hence, by Theorem I.3.1 in [18], there exists a stream function $\chi \in H^1(\Omega)$ such that $\boldsymbol{\tau} - \boldsymbol{\Phi} - \frac{d}{2}(x_1-a,x_2-b)^t = \mathbf{curl}(\chi)$ in $\Omega$. In addition, we have

$$\begin{aligned}
|\chi|^2_{H^1(\Omega)} &= ||\mathbf{curl}(\chi)||^2_{L^2(\Omega)} = \left\|\boldsymbol{\tau} - \boldsymbol{\Phi} - \frac{d}{2}(x_1-a,x_2-b)^{\mathrm{t}}\right\|^2_{L^2(\Omega)} \\
&\leq 2\left(||\boldsymbol{\tau}||^2_{[L^2(\Omega)]^2} + ||\boldsymbol{\Phi}||^2_{[L^2(\Omega)]^2} + \frac{d^2}{4}||(x_1-a,x_2-b)||^2_{[L^2(\Omega)]^2}\right) \\
&\leq 2\left(||\boldsymbol{\tau}||^2_{[L^2(\Omega)]^2} + ||\boldsymbol{\Phi}||^2_{[L^2(\Omega)]^2} + \frac{d^2}{4}\big(\mathrm{diam}(\Omega)\big)^2\,|\Omega|\right) \\
&\leq 2\,\max\left\{1, c^2 + \frac{1}{4}\big(\mathrm{diam}(\Omega)\big)^2\right\}\,||\boldsymbol{\tau}||^2_{H(\mathrm{div};\Omega)}\,.
\end{aligned}$$

As a result, we establish (3.11), and we end the proof. □

Now, considering $\chi$ and $\boldsymbol{\Phi}$ as the ones provided by Lemma 3.3 for a given $\boldsymbol{\tau} \in H(\mathrm{div};\Omega)$, we introduce $\chi_h := I_h(\chi)$, and define

$$\boldsymbol{\tau}_h := \mathbf{curl}(\chi_h) + \pi_h^r(\boldsymbol{\Phi}) + \frac{d}{2}\begin{pmatrix} x_1-a \\ x_2-b\end{pmatrix} \in H_h^{\boldsymbol{\sigma}}\,, \tag{3.12}$$

which is referred as a *discrete quasi-Helmholtz decomposition* of $\boldsymbol{\tau}_h$. Therefore, we can write

$$\boldsymbol{\tau} - \boldsymbol{\tau}_h = \mathbf{curl}(\chi - \chi_h) + \boldsymbol{\Phi} - \pi_h^r(\boldsymbol{\Phi})\,, \tag{3.13}$$

that verifies

$$\mathrm{div}(\boldsymbol{\tau} - \boldsymbol{\tau}_h) = \mathrm{div}(\boldsymbol{\Phi} - \pi_h^r(\boldsymbol{\Phi})) \tag{3.14}$$

On the other hand, it is not difficult to check the following orthogonality relation

$$A((\boldsymbol{\sigma} - \boldsymbol{\sigma}_h, u-u_h),(\boldsymbol{\zeta}_h, v_h)) = 0\,, \quad \forall\,(\boldsymbol{\zeta}_h, v_h) \in \boldsymbol{\Sigma}_h := H_h^{\boldsymbol{\sigma}} \times H_h^u\,. \tag{3.15}$$

From now on, given $(\boldsymbol{\tau}, v) \in \boldsymbol{\Sigma}$, we associate it with the discrete pair $(\boldsymbol{\tau}_h, 0) \in \boldsymbol{\Sigma}_h$, where $\boldsymbol{\tau}_h$ is defined as in (3.12). Hence, considering (3.15) with $(\boldsymbol{\zeta}_h, v_h) := (\boldsymbol{\tau}_h, 0)$, and knowing that $(\boldsymbol{\sigma}, u)$ is the unique solution of problem (2.1), we obtain

$$\begin{aligned}
\langle(\bar{\boldsymbol{\sigma}},\bar{\mathbf{u}}),(\boldsymbol{\tau},\mathbf{v})\rangle_{\boldsymbol{\Sigma}} &= A((\boldsymbol{\sigma} - \boldsymbol{\sigma}_h, u-u_h),(\boldsymbol{\tau} - \boldsymbol{\tau}_h, v)) \\
&= -\,\langle(\boldsymbol{\tau} - \boldsymbol{\tau}_h)\cdot\boldsymbol{n}, g\rangle - \int_\Omega f\,v - A((\boldsymbol{\sigma}_h, u_h),(\boldsymbol{\tau} - \boldsymbol{\tau}_h, v))\,.
\end{aligned}$$

Equivalently,

$$(\bar{\boldsymbol{\sigma}}, \boldsymbol{\tau})_{H(\mathrm{div};\Omega)} = F_1(\boldsymbol{\tau} - \boldsymbol{\tau}_h) \quad \forall\, \boldsymbol{\tau} \in H(\mathrm{div};\Omega)\,,$$
$$(\bar{u}, v)_{L^2(\Omega)} = F_2(v) \quad \forall\, v \in L^2(\Omega)\,,$$

where $F_1 : H(\mathrm{div};\Omega) \to \mathbb{R}$ and $F_2 : L^2(\Omega) \to \mathbb{R}$ are the bounded linear functionals defined by

$$F_1(\boldsymbol{\rho}) := -\langle \boldsymbol{\rho}\cdot\boldsymbol{n}, g\rangle - \int_\Omega \boldsymbol{\sigma}_h \cdot \boldsymbol{\rho} + \int_\Omega u_h \,\mathrm{div}(\boldsymbol{\rho})\,, \quad \forall\, \boldsymbol{\rho} \in H(\mathrm{div};\Omega)\,,$$
$$F_2(w) := -\int_\Omega (f - \mathrm{div}(\boldsymbol{\sigma}_h))\, w\,, \quad \forall w \in L^2(\Omega)\,.$$

Hence, taking into account (3.13) and (3.14), and the fact that $\pi_h^k(\boldsymbol{\Phi})\cdot\boldsymbol{n} = 0$ on $\Gamma$, we can rewrite $F_1(\boldsymbol{\tau} - \boldsymbol{\tau}_h)$ as follows

$$F_1(\boldsymbol{\tau} - \boldsymbol{\tau}_h) = R_1(\boldsymbol{\Phi}) + R_2(\chi)\,,$$

where

$$R_1(\boldsymbol{\Phi}) := -\int_\Omega (\boldsymbol{\sigma}_h + \nabla_h u_h)\cdot(\boldsymbol{\Phi} - \pi_h^k(\boldsymbol{\Phi})) + \sum_{T\in\mathscr{T}_h} \int_{\partial T \cap E_I} u_h\,(\boldsymbol{\Phi} - \pi_h^k(\boldsymbol{\Phi}))\cdot\boldsymbol{n}\,,$$
$$R_2(\chi) := -\langle \mathbf{curl}(\chi - \chi_h)\cdot\boldsymbol{n}, g\rangle - \int_\Omega \boldsymbol{\sigma}_h \cdot \mathbf{curl}(\chi - \chi_h)\,.$$

Our aim now is to obtain upper bounds for each one of the terms $F_2(v)$, $R_1(\boldsymbol{\Phi})$ and $R_2(\chi)$.

**Lemma 3.4.** *For any $v \in L^2(\Omega)$ there holds*

$$|F_2(v)| \leq \left(\sum_{T\in\mathscr{T}_h} \|f - \mathrm{div}(\boldsymbol{\sigma}_h)\|^2_{L^2(T)}\right)^{1/2} \|v\|_{L^2(\Omega)}\,.$$

**Proof.** The proof follows from a straightforward application of Cauchy-Schwarz inequality. □

**Lemma 3.5.** *There exists $C > 0$, independent of h, such that*

$$|R_1(\boldsymbol{\Phi})| \leq C\left(\sum_{e\in E_I} h_e \,\|[\![u_h]\!]\|^2_{[L^2(e)]^2} + \sum_{T\in\mathscr{T}_h} h_T^2\,\|\nabla u_h + \boldsymbol{\sigma}_h\|^2_{[L^2(T)]^2}\right)^{1/2} \|\boldsymbol{\tau}\|_{H(\mathrm{div};\Omega)}\,.$$

**Proof.** It is a slight modification of Lemma 3.5 in [5]. We omit further details. □

**Lemma 3.6.** *Under the assumption that $g \in H^1(\Gamma)$, there exists $C > 0$, independent of h, such that*

$$|R_2(\chi)| \leq C\Bigg(\sum_{T\in\mathscr{T}_h} h_T^2 \|\mathrm{rot}(\boldsymbol{\sigma}_h)\|^2_{L^2(T)} + \sum_{e\in E(T)} h_e \left(\|[\![\boldsymbol{\sigma}_h\cdot\boldsymbol{t}]\!]\|^2_{L^2(e\cap E_I)} + \left\|\boldsymbol{\sigma}_h\cdot\boldsymbol{t} + \frac{dg}{d\boldsymbol{t}}\right\|^2_{L^2(e\cap E_\Gamma)}\right)\Bigg)^{1/2} \|\boldsymbol{\tau}\|_{H(\mathrm{div};\Omega)}\,.$$

**Proof.** Knowing that $\mathbf{curl}(\chi-\chi_h)\cdot\boldsymbol{n} = \frac{d}{d\boldsymbol{t}}(\chi-\chi_h)$ on $\Gamma$, and after integrating by parts, we deduce

$$\begin{aligned}
R_2(\chi) &= \langle \mathbf{curl}(\chi-\chi_h)\cdot\boldsymbol{n}, g\rangle + \int_\Omega \boldsymbol{\sigma}_h \cdot \mathbf{curl}(\chi-\chi_h)\\
&= \left\langle \frac{d}{d\boldsymbol{t}}(\chi-\chi_h), g\right\rangle + \sum_{T\in\mathscr{T}_h}\int_T \boldsymbol{\sigma}_h\cdot\mathbf{curl}(\chi-\chi_h)\\
&= -\left\langle \chi-\chi_h, \frac{dg}{d\boldsymbol{t}}\right\rangle + \sum_{T\in\mathscr{T}_h}\left(\int_T \mathrm{rot}(\boldsymbol{\sigma}_h)\,(\chi-\chi_h) - \langle \chi-\chi_h, \boldsymbol{\sigma}_h\cdot\boldsymbol{t}\rangle_{\partial T}\right)\\
&= \sum_{T\in\mathscr{T}_h}\int_T \mathrm{rot}(\boldsymbol{\sigma}_h)\,(\chi-\chi_h) + \int_{E_I}(\chi-\chi_h)\,[\![\boldsymbol{\sigma_h}\cdot\boldsymbol{t}]\!] - \int_{E_\Gamma}(\chi-\chi_h)\left(\boldsymbol{\sigma}_h\cdot\boldsymbol{t}+\frac{dg}{d\boldsymbol{t}}\right).
\end{aligned}$$

$$\begin{aligned}
\Rightarrow |R_2(\chi)| \le &\sum_{T\in\mathscr{T}_h}\|\mathrm{rot}(\boldsymbol{\sigma}_h)\|_{L^2(T)}\|\chi-\chi_h\|_{L^2(T)} + \sum_{e\in E_I}\|\chi-\chi_h\|_{L^2(e)}\|[\![\boldsymbol{\sigma}_h\cdot\boldsymbol{t}]\!]\|_{L^2(e)}\\
&+\sum_{e\in E_\Gamma}\|\chi-\chi_h\|_{L^2(e)}\left\|\boldsymbol{\sigma}_h\cdot\boldsymbol{t}+\frac{dg}{d\boldsymbol{t}}\right\|_{L^2(e)}.
\end{aligned}$$

Therefore, the proof is completed invoking Lemma 3.1, the Cauchy-Schwarz inequality, the regularity of the mesh and (3.11). □

The previous results suggest the definition of the following residual estimator

$$\eta := \left(\sum_{T\in\mathscr{T}_h}\eta_T^2\right)^{1/2}, \tag{3.16}$$

where

$$\begin{aligned}
\eta_T^2 := &\|f-\mathrm{div}(\boldsymbol{\sigma}_h)\|^2_{L^2(T)} + h_T^2\|\boldsymbol{\sigma}_h+\nabla u_h\|^2_{[L^2(T)]^2} + h_T^2\|\mathrm{rot}(\boldsymbol{\sigma}_h)\|^2_{L^2(T)}\\
&+\sum_{e\in E(T)} h_e\left(\|[\![u_h]\!]\|^2_{L^2(e\cap E_I)} + \|[\![\boldsymbol{\sigma}_h\cdot\boldsymbol{t}]\!]\|^2_{L^2(e\cap E_I)} + \left\|\boldsymbol{\sigma}_h\cdot\boldsymbol{t}+\frac{dg}{d\boldsymbol{t}}\right\|^2_{L^2(e\cap E_\Gamma)}\right).
\end{aligned}$$

An upper bound for $||(\bar{\boldsymbol{\sigma}},\bar{u})||_{\boldsymbol{\Sigma}}$ is established in the next lemma, in terms of (3.16).

**Lemma 3.7.** *Assuming that $g\in H^1(\Gamma)$, there exists a constant $C>0$, independent of $h$, such that*

$$||(\bar{\boldsymbol{\sigma}},\bar{u})||_{\boldsymbol{\Sigma}} \le C\,\eta\,. \tag{3.17}$$

**Proof.** Invoking Lemmas 3.5 and 3.6, we deduce that there exists $C>0$, independent of $h$, such that

$$\begin{aligned}
|F_1(\boldsymbol{\tau}-\boldsymbol{\tau}_h)| \le C\Bigg(&\sum_{T\in\mathscr{T}_h} h_T^2\|\boldsymbol{\sigma}_h+\nabla u_h\|^2_{[L^2(T)]^2} + \sum_{e\in E_h} h_e\Bigg(\|[\![\boldsymbol{\sigma}_h\cdot\boldsymbol{t}]\!]\|^2_{L^2(e\cap E_I)}\\
&+\left\|\boldsymbol{\sigma}_h\cdot\boldsymbol{t}+\frac{dg}{d\boldsymbol{t}}\right\|^2_{L^2(e\cap E_\Gamma)} + \|[\![u_h]\!]\|^2_{L^2(e\cap E_I)}\Bigg)\Bigg)^{1/2}\|\boldsymbol{\tau}\|_{H(\mathrm{div};\Omega)}\,.
\end{aligned}$$

Hence, (3.17) follows from the above bound, Lemma 3.4 and a discrete Cauchy-Schwarz inequality. □

The following theorem establishes the main result of this section, which is the reliability and efficiency of the estimator $\eta$.

**Theorem 3.2.** *There exists a positive constant $C_{\mathtt{rel}}$, independent of h, such that*

$$||(\boldsymbol{\sigma}-\boldsymbol{\sigma}_h, u-u_h)||_{\boldsymbol{\Sigma}} \leq C_{\mathtt{rel}}\, \eta\,. \tag{3.18}$$

*Additionally, there exists $C_{\mathtt{eff}} > 0$, independent of h, such that*

$$\eta_T^2 \leq C_{\mathtt{eff}}\, ||(\boldsymbol{\sigma}-\boldsymbol{\sigma}_h, u-u_h)||_{\boldsymbol{\Sigma}(T)}\,, \tag{3.19}$$

*where $\forall\, T \in \mathscr{T}_h\, :\, ||(\boldsymbol{\tau}, v)||^2_{\boldsymbol{\Sigma}(T)} := \|\boldsymbol{\tau}\|^2_{H(\mathrm{div};T)} + \|v\|^2_{L^2(T)}$.* **Proof.** The reliability of $\eta$, (3.18), follows from (3.9) and Lemma 3.7. The efficiency of $\eta$, (3.19), is treated in the next subsection. We omit further details. □

### 3.3 Efficiency of the estimator

In this subsection we prove the local efficiency of the estimator $\eta$ (cf. (3.19)). We begin by introducing some notations and preliminary results. Given $T \in \mathscr{T}_h$ and $e \in E(T)$, we let $\psi_T$ and $\psi_e$ be the standard triangle-bubble and edge-bubble functions, respectively. In particular, $\psi_T$ satisfies $\psi_T \in \mathscr{P}_3(T)$, $\mathrm{supp}(\psi_T) \subseteq T$, $\psi_T = 0$ on $\partial T$, and $0 \leq \psi_T \leq 1$ in $T$. Similarly, $\psi_e|_T \in \mathscr{P}_2(T)$, $\mathrm{supp}(\psi_e) \subseteq \omega_e := \cup\{T' \in \mathscr{T}_h : \quad e \in E(T')\}$, $\psi_e = 0$ on $\partial\omega_e$, and $0 \leq \psi_e \leq 1$ in $\omega_e$. We also recall from [21] that, given $k \in \mathbb{N}\cup\{0\}$, there exists an extension operator $L : C(e) \to C(T)$ that satisfies $L(p) \in \mathscr{P}_k(T)$ and $L(p)|_e = p\ \forall\, p \in \mathscr{P}_k(e)$. Additional properties of $\psi_T$, $\psi_e$, and $L$ are collected in the following lemma.

**Lemma 3.8.** *For any triangle $T$ there exist positive constants $c_1$, $c_2$, $c_3$ and $c_4$, depending only on k and the shape of T, such that for all $q \in \mathscr{P}_k(T)$ and $p \in \mathscr{P}_k(e)$, there hold*

$$||\psi_T\, q||^2_{L^2(T)} \leq ||q||^2_{L^2(T)} \leq c_1\, ||\psi_T^{1/2}\, q||^2_{L^2(T)}\,, \tag{3.20}$$

$$||\psi_e\, p||^2_{L^2(e)} \leq ||p||^2_{L^2(e)} \leq c_2\, ||\psi_e^{1/2}\, p||^2_{L^2(e)}\,, \tag{3.21}$$

$$c_4\, h_e\, ||p||^2_{L^2(e)} \leq ||\psi_e^{1/2}\, L(p)||^2_{L^2(T)} \leq c_3\, h_e\, ||p||^2_{L^2(e)}\,, \tag{3.22}$$

**Proof.** See Lemma 4.1 in [21]. □

The following inverse estimate will also be useful.

**Lemma 3.9.** *Let $\ell, m \in \mathbb{N}\cup\{0\}$ such that $\ell \leq m$. Then, for any triangle $T$, there exists $c > 0$, depending only on $k, \ell, m$ and the shape of T, such that*

$$|q|_{H^m(T)} \leq c\, h_T^{\ell-m} |q|_{H^\ell(T)} \quad \forall\, q \in \mathscr{P}_k(T)\,. \tag{3.23}$$

**Proof.** See Theorem 3.2.6 in [14]. □

Since $f = \operatorname{div}(\boldsymbol{\sigma})$ in $\Omega$, we have that

$$||f - \operatorname{div}(\boldsymbol{\sigma}_h)||_{L^2(T)} = ||\operatorname{div}(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h)||_{L^2(T)}\,.$$

**Lemma 3.10.** *There exists $C_1 > 0$, independent of the meshsize, such that for any $T \in \mathscr{T}_h$*

$$h_T\,||\boldsymbol{\sigma}_h + \nabla u_h||_{[L^2(T)]^2} \le C_1\Big(||u - u_h||_{L^2(T)} + h_T\,||\boldsymbol{\sigma} - \boldsymbol{\sigma}_h||_{[L^2(T)]^2}\Big)\,.$$

**Proof.** We introduce $\boldsymbol{\rho}_h := \boldsymbol{\sigma}_h + \nabla u_h$ in $T$. Then, taking into account the property (3.20) and integrating by parts, we have

$$\begin{aligned}
& c_1^{-1}\,||\boldsymbol{\rho}_h||^2_{[L^2(T)]^2} \le ||\psi_T^{1/2}\boldsymbol{\rho}_h||^2_{[L^2(T)]^2} = \int_T (\boldsymbol{\sigma}_h + \nabla u_h)\cdot \psi_T \boldsymbol{\rho}_h \\
&= \int_T \boldsymbol{\sigma}_h \cdot \psi_T \boldsymbol{\rho}_h + \int_T \nabla u_h \cdot \psi_T \boldsymbol{\rho}_h = \int_T \boldsymbol{\sigma}_h \cdot \psi_T \boldsymbol{\rho}_h - \int_T u_h \operatorname{div}(\psi_T \boldsymbol{\rho}_h) \\
&= \int_T (\boldsymbol{\sigma}_h - \boldsymbol{\sigma})\cdot \psi_T \boldsymbol{\rho}_h + \int_T (u - u_h)\operatorname{div}(\psi_T \boldsymbol{\rho}_h)\,.
\end{aligned}$$

Now, applying Cauchy-Schwarz inequality as well as inverse inequality (3.23) and property $0 \le \psi_T \le 1$, we derive

$$\begin{aligned}
& c_1^{-1}\,||\boldsymbol{\rho}_h||^2_{[L^2(T)]^2} \le \Big\{||\boldsymbol{\sigma} - \boldsymbol{\sigma}_h||_{[L^2(T)]^2}||\psi_T^{1/2}\boldsymbol{\rho}_h||_{[L^2(T)]^2} \\
& + ||u - u_h||_{L^2(T)}\|\operatorname{div}(\psi_T \boldsymbol{\rho}_h)\|_{[L^2(T)]^2}\Big\} \\
& \le ||\boldsymbol{\sigma} - \boldsymbol{\sigma}_h||_{[L^2(T)]^2}\,||\boldsymbol{\rho}_h||_{[L^2(T)]^2} + \sqrt{2}||u - u_h||_{L^2(T)}\,||\nabla(\psi_T \boldsymbol{\rho}_h)||_{[L^2(T)]^{2\times 2}} \\
& \le C\Big\{||\boldsymbol{\sigma} - \boldsymbol{\sigma}_h||_{[L^2(T)]^2} + \sqrt{2}h_T^{-1}||u - u_h||_{L^2(T)}\Big\}\,||\boldsymbol{\rho}_h||_{[L^2(T)]^2}\,.
\end{aligned}$$

Hence, simplifying $||\boldsymbol{\rho}_h||_{[L^2(T)]^2}$ and multiplying by the factor $h_T$, we complete the proof of the lemma. □

In the following lemma, we bound the jump of $u_h$,

**Lemma 3.11.** *There exists $C_2 > 0$, independent of the mesh size, such that for any $e \in E_I$*

$$h_e||[\![u_h]\!]||^2_{L^2(e)} \le C_2\left\{||u - u_h||^2_{L^2(\omega_e)} + ||\boldsymbol{\sigma} - \boldsymbol{\sigma}_h||^2_{[L(\omega_e)]^2}\right\}\,. \tag{3.24}$$

**Proof.** First, given $e \in E_I$ we set $\omega_e := T \cup T'$, with $T, T' \in \mathscr{T}_h$ such that $e = \partial T \cap \partial T'$. Next, we introduce $w_h := [\![u_h]\!]$ on $e$ and $\boldsymbol{\rho}_e := \psi_e L(w_h)\boldsymbol{n}_{T,e}$ in $\omega_e$, which belongs to $H(\operatorname{div}, \omega_e)$. Taking into account (3.21), knowing that $[\![u]\!] = 0$ on $E_I$, and integrating by parts, we derive

$$\begin{aligned}
c_2^{-1}||w_h||^2_{L^2(e)} \le ||\psi_e^{1/2}w_h||^2_{L^2(e)} &= \int_e \psi_e L(w_h)[\![u_h - u]\!] = \int_e [\![u_h - u]\!]\boldsymbol{\rho}_e\cdot \boldsymbol{n}_T \\
&= \int_{\omega_e}(u_h - u)\operatorname{div}(\boldsymbol{\rho}_e) + \int_{\omega_e}\nabla_h(u_h - u)\cdot \boldsymbol{\rho}_e \\
= \int_{\omega_e}(u_h - u)\operatorname{div}(\boldsymbol{\rho}_e) &+ \int_{\omega_e}(\boldsymbol{\sigma}_h + \nabla_h u_h)\cdot \boldsymbol{\rho}_e + \int_{\omega_e}(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h)\cdot \boldsymbol{\rho}_e\,.
\end{aligned}$$

Using the fact that $\int_{\omega_e} = \int_T + \int_{T'}$ and applying Cauchy-Schwarz inequality, we deduce

$$\begin{aligned} c_2^{-1}||w_h||^2_{L^2(e)} & \\ \leq \|u-u_h\|_{L^2(T)}\|\mathrm{div}(\boldsymbol{\rho}_e)\|_{L^2(T)} + \|u-u_h\|_{L^2(T')}\|\mathrm{div}(\boldsymbol{\rho}_e)\|_{L^2(T')} & \\ + \|\boldsymbol{\sigma}_h + \nabla u_h\|_{[L^2(T)]^2}\|\boldsymbol{\rho}_e\|_{[L^2(T)]^2} + \|\boldsymbol{\sigma}_h + \nabla u_h\|_{[L^2(T')]^2}\|\boldsymbol{\rho}_e\|_{[L^2(T')]^2} & \qquad (3.25) \\ + \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{[L^2(T)]^2}\|\boldsymbol{\rho}_e\|_{[L^2(T)]^2} + \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{[L^2(T')]^2}\|\boldsymbol{\rho}_e\|_{[L^2(T')]^2}\,. & \end{aligned}$$

Now, invoking the inverse inequality (3.23) and knowing that $0 \leq \psi_e \leq 1$ in $\omega_e$ together with (3.22), we arrive for each $T \in \omega_e$

$$\begin{aligned} \|\mathrm{div}(\boldsymbol{\rho}_e)\|_{L^2(T)} &\leq \sqrt{2}\|\nabla\boldsymbol{\rho}_e\|_{[L^2(T)]^{2\times 2}} \leq c\sqrt{2}h_T^{-1}\|\boldsymbol{\rho}_e\|_{[L^2(T)]^2} \\ &= c\sqrt{2}h_T^{-1}\|\psi_e^{1/2}L(w_h)\|_{L^2(T)} \leq c\,c_3\sqrt{2}h_T^{-1/2}\|w_h\|_{L^2(e)}\,. \end{aligned}$$

This inequality, together with (3.22), allow us to rewrite (3.25) as follows: There exists $c > 0$ independent of mesh size, such that

$$\begin{aligned} c||w_h||^2_{L^2(e)} \leq \Big\{ & h_T^{-1/2}\|u-u_h\|_{L^2(T)} + h_{T'}^{-1/2}\|u-u_h\|_{L^2(T')} \\ & + h_T\ \|\boldsymbol{\sigma}_h + \nabla u_h\|_{[L^2(T)]^2} + h_{T'}\|\boldsymbol{\sigma}_h + \nabla u_h\|_{[L^2(T')]^2} \\ & + h_T\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{[L^2(T)]^2} + h_{T'}\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{[L^2(T')]^2}\Big\}\|w_h\|_{L^2(e)}\,. \end{aligned}$$

Then the proof follows after multiplying by $h_e$, and applying Lemma 3.10. □

**Lemma 3.12.** *There exists $C_3 > 0$, independent of the meshsize, such that for any $T \in \mathscr{T}_h$*

$$h_T\,||\mathrm{rot}(\boldsymbol{\sigma}_h)||_{L^2(T)} \leq C_3\,||\boldsymbol{\sigma} - \boldsymbol{\sigma}_h||_{[L^2(T)]^2}\,.$$

**Proof.** We introduce $\rho_h := \mathrm{rot}(\boldsymbol{\sigma}_h)$ in $T$. Then, invoking the property (3.20), $\mathrm{rot}(\boldsymbol{\sigma}) = 0$ in $T$, and integrating by parts, we have

$$\begin{aligned} c_1^{-1}\,||\rho_h||^2_{L^2(T)} &\leq ||\psi_T^{1/2}\rho_h||^2_{[L^2(T)]^2} = \int_T \mathrm{rot}(\boldsymbol{\sigma}_h)\psi_T\rho_h \\ &= \int_T \mathrm{rot}(\boldsymbol{\sigma}_h - \boldsymbol{\sigma})\psi_T\rho_h = \int_T (\boldsymbol{\sigma}_h - \boldsymbol{\sigma})\cdot\mathbf{curl}(\psi_T\rho_h)\,. \end{aligned}$$

Now, applying Cauchy-Schwarz inequality, as well as inverse inequality (3.23) and the fact that $0 \leq \psi_T \leq 1$ in $T$, we derive

$$\begin{aligned} c_1^{-1}\,||\rho_h||^2_{[L^2(T)]^2} &\leq ||\boldsymbol{\sigma} - \boldsymbol{\sigma}_h||_{[L^2(T)]^2}||\mathbf{curl}(\psi_T\rho_h)||_{[L^2(T)]^2} \\ &= ||\boldsymbol{\sigma} - \boldsymbol{\sigma}_h||_{[L^2(T)]^2}\,||\nabla(\psi_T\rho_h)||_{[L^2(T)]^2} \\ &\leq C\,||\boldsymbol{\sigma} - \boldsymbol{\sigma}_h||_{[L^2(T)]^2}h_T^{-1}\,||\psi_T\rho_h||_{L^2(T)} \\ &\leq C\,h_T^{-1}||\boldsymbol{\sigma} - \boldsymbol{\sigma}_h||_{[L^2(T)]^2}\,||\rho_h||_{L^2(T)}\,. \end{aligned}$$

Hence, simplifying $||\rho_h||_{[L^2(T)]^2}$ and multiplying by the factor $h_T$, we complete the proof of the lemma. □

The tangential component jump of $\boldsymbol{\sigma}_h$ is treated in the next lemma.

**Lemma 3.13.** *There exists $C_4 > 0$, independent of the mesh size, such that for any $e \in \mathscr{E}_I$*

$$h_e||[\![\boldsymbol{\sigma}_h \cdot \boldsymbol{t}]\!]||^2_{L^2(e)} \le C_4 ||\boldsymbol{\sigma} - \boldsymbol{\sigma}_h||^2_{[L(\omega_e)]^2}\,. \tag{3.26}$$

**Proof.** Given $e \in E_I$, let $T, T' \in \mathscr{T}_h$ such that $\omega_e = T \cup T'$ and they share $e$, i.e. $\partial T \cap \partial T' = e$. Denoting by $w_h := [\![\boldsymbol{\sigma}_h \cdot \boldsymbol{t}]\!]$ on $e$, and using 3.21, it follows that

$$\begin{aligned} c_2^{-1}||w_h||^2_{L^2(e)} &\le ||\psi_e^{1/2} w_h||^2_{L^2(e)} = \int_e \psi_e L(w_h)\, [\![\boldsymbol{\sigma}_h \cdot \boldsymbol{t}]\!] \\ &= \int_e \psi_e L(w_h)\, \boldsymbol{\sigma}_h \cdot \boldsymbol{t}_T + \int_e \psi_e L(w_h)\, \boldsymbol{\sigma}_h \cdot \boldsymbol{t}_{T'} \\ &= -\int_{\omega_e} \mathbf{curl}(\psi_e L(w_h)) \cdot \boldsymbol{\sigma}_h + \int_{\omega_e} \psi_e L(w_h)\, \mathrm{rot}(\boldsymbol{\sigma}_h) \\ &= \int_{\omega_e} \mathbf{curl}(\psi_e L(w_h)) \cdot (\boldsymbol{\sigma} - \boldsymbol{\sigma}_h) + \int_{\omega_e} \psi_e L(w_h)\, \mathrm{rot}_h(\boldsymbol{\sigma}_h)\,, \end{aligned} \tag{3.27}$$

where in the last equality we take into account

$$\int_{\omega_e} \mathbf{curl}(\psi_e L(w_h)) \cdot \boldsymbol{\sigma} = -\int_{\omega_e} \mathbf{curl}(\psi_e L(w_h)) \cdot \nabla u = \int_{\partial\omega_e} \psi_e L(w_h)\, \nabla u \cdot \boldsymbol{t} = 0\,.$$

In addition, realizing that $\int_{\omega_e} = \int_T + \int_{T'}$ and applying Cauchy-Schwarz inequality, we deduce

$$\begin{aligned} c_2^{-1}||w_h||^2_{L^2(e)} \le\; & \|\mathbf{curl}(\psi_e L(w_h))\|_{[L^2(T)]^2} \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{[L^2(T)]^2} \\ & + \|\psi_e L(w_h)\|_{L^2(T)} \|\mathrm{rot}(\boldsymbol{\sigma}_h)\|_{L^2(T)} \\ & + \|\mathbf{curl}(\psi_e L(w_h))\|_{[L^2(T')]^2} \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{[L^2(T')]^2} \\ & + \|\psi_e L(w_h)\|_{L^2(T')} \|\mathrm{rot}(\boldsymbol{\sigma}_h)\|_{L^2(T')}\,. \end{aligned} \tag{3.28}$$

Now, knowing that $0 \le \psi_e^{1/2} \le 1$, and taking into account (3.22), for each $T \in \mathscr{T}_h$, we deduce

$$\|\psi_e L(w_h)\|_{L^2(T)} \le c_3 h_T^{1/2} \|w_h\|_{L^2(e)}\,. \tag{3.29}$$

Now, the inverse inequality (3.23)), the fact that $0 \le \psi_e^{1/2} \le 1$ in $\omega_e$, together with (3.22)), implies for each $T \in \omega_e$

$$\begin{aligned} \|\mathbf{curl}(\psi_e L(w_h))\|_{[L^2(T)]^2} &= \|\nabla(\psi_e L(w_h))\|_{[L^2(T)]^2} \\ &\le c h_T^{-1} \|\psi_e L(w_h)\|_{L^2(T)} \le c h_T^{-1} \|\psi_e^{1/2} L(w_h)\|_{L^2(T)} \\ &\le c\, c_3 h_T^{-1/2} \|w_h\|_{L^2(e)}\,. \end{aligned} \tag{3.30}$$

Inequalities (3.29) and (3.30) allow us to rewrite (3.28) as follows: There exists $c > 0$ independent of meshsize, such that

$$c||w_h||^2_{L^2(e)} \leq \Big\{ h_T^{-1/2} \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{[L^2(T)]^2} + h_{T'}^{-1/2} \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{[L^2(T')]^2} + h_T^{1/2} \, \|\mathrm{rot}(\boldsymbol{\sigma}_h)\|_{L^2(T)} + h_{T'}^{1/2} \|\mathrm{rot}(\boldsymbol{\sigma}_h)\|_{L^2(T')} \Big\} \|w_h\|_{L^2(e)} \,.$$

Then, (3.26) follows after simplifying $||w_h||_{L^2(e)}$, multiplying by $h_e^{1/2}$ and invoking Lemma 3.12. □

**Remark 3.14.** *The current a posteriori error analysis can be extended to three dimensions. To this aim, we consider $\Omega$ a bounded and simply connected polyhedral domain in $\mathbb{R}^3$. Now, given a partition $\mathscr{T}_h$ of $\overline{\Omega}$ made of tetrahedral, we take into account similar notations as the ones introduced in Section 3, with face instead of edge. In addition, for any smooth enough vector field $\boldsymbol{\rho}$, respectively, we set $\mathbf{curl}(\boldsymbol{\rho}) := \nabla \times \boldsymbol{\rho}$, while the jump of tangential trace of $\boldsymbol{\rho}$ across $e \in E_h$, by*

$$[\![\boldsymbol{n} \times \boldsymbol{\rho}]\!] := \begin{cases} \boldsymbol{n}_{T,e} \times \boldsymbol{\rho}_{T,e} + \boldsymbol{n}_{T',e} \times \boldsymbol{\rho}_{T',e} & e \in E_I \,, \\ \boldsymbol{n}_{T,e} \times \boldsymbol{\rho}_{T,e} & e \in E_\Gamma \,, \end{cases}$$

*where $T, T' \in \mathscr{T}_h$ are the pair of tetrahedral sharing the face $e \in E_I$. On the other hand, when $e \in E_\Gamma$, by $T \in \mathscr{T}_h$ we refer to the unique element having e as a boundary face. Now, following the ideas given in the proof of Lemma 3.3, and applying Theorem I.3.5 in [18], we can establish the 3D-version of the quasi-Helmholtz decomposition of functions belonging to $H(\mathbf{div};\Omega)$ presented in Lemma 3.3, which in addition is also stable (invoking Theorem 2.1 in [13]). This means that for any $\boldsymbol{\tau} \in H(\mathrm{div};\Omega)$, there exist $\boldsymbol{\chi} \in [H^1(\Omega)]^3$ and $\boldsymbol{\Phi} \in [H^1_0(\Omega)]^3$, such that*

$$\boldsymbol{\tau} = \mathbf{curl}(\boldsymbol{\chi}) + \boldsymbol{\Phi} + \frac{d}{3} \begin{pmatrix} x_1 - a \\ x_2 - b \\ x_3 - c \end{pmatrix},$$

*where $(a,b,c)^{\mathrm{t}}$ is any fixed point belonging to $\Omega$, and $d := \frac{1}{|\Omega|} \int_\Omega \mathrm{div}(\boldsymbol{\tau})$. In addition, there exists $C > 0$, such that*

$$|\boldsymbol{\chi}|^2_{[H^1(\Omega)]^3} + ||\boldsymbol{\Phi}||^2_{[H^1(\Omega)]^3} \leq C\,||\boldsymbol{\tau}||^2_{H(\mathrm{div};\Omega)} \,.$$

*Then, proceeding in analogous way as in Section 3, we prove a similar result to Theorem 3.2, where the local a posteriori error estimator now reads as*

$$\eta_T^2 := ||f + \mathrm{div}(\boldsymbol{\sigma}_h)||^2_{[L^2(T)]^3} + h_T^2 \, \|\boldsymbol{\sigma}_h + \nabla \boldsymbol{u}_h\|^2_{[L^2(T)]^3} + h_T^2 \, \|\mathbf{curl}(\boldsymbol{\sigma}_h)\|^2_{[L^2(T)]^3} + \sum_{e \in E(T) \cap E_I} \Big\{ h_e \, \|[\![\boldsymbol{u}_h]\!]\|^2_{[L^2(e)]^3} + h_e \, \|[\![\boldsymbol{n} \times \boldsymbol{\sigma}_h]\!]\|^2_{[L^2(e)]^3} \Big\} + \sum_{e \in E(T) \cap E_\Gamma} h_e \, \|\boldsymbol{n} \times (\boldsymbol{\sigma}_h + \nabla \boldsymbol{g})\|^2_{[L^2(e)]^3} \,. \tag{3.31}$$

## 4 NUMERICAL EXAMPLE

In this section, we present one numerical example illustrating the performance of the dual mixed method when applied to the Poisson problem, with Dirichlet condition, as well as of the corresponding adaptive procedure. We consider the lowest finite element $\mathcal{RT}_0(T) - \mathcal{P}_0(T)$ for our approximation. We remark that the computational implementation has been done using a MATLAB code.

Hereafter, the number of degrees of freedom (unknowns) is given by $N :=$ number of edges + number of elements, induced by the triangulation. Moreover, the involved individual and total errors are defined as $e_0(u) := \|u - u_h\|_{L^2(\Omega)}$, $e(\boldsymbol{\sigma}) := \left(\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|^2_{[L^2(\Omega)]^2} + \|\mathrm{div}(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h)\|^2_{L^2(\Omega)}\right)^{1/2}$ and $e := (e_0(u)^2 + e(\boldsymbol{\sigma})^2)^{1/2}$, where $(\boldsymbol{\sigma}, u) \in H(\mathrm{div};\Omega) \times L^2(\Omega)$ and $(\boldsymbol{\sigma}_h, u_h) \in H^{\boldsymbol{\sigma}}_{h,r} \times H^u_{h,r}$ are the corresponding unique solutions of the continuous (2.1) and discrete (2.2) formulations. Additionally, if $e$ and $e'$ stand for the errors at two consecutive triangulations with $N$ and $N'$ number of degrees of freedom, respectively, we set the experimental rate of convergence of the global error as $r := -2\dfrac{\log(e/e')}{\log(N/N')}$. We define $r_0(u)$ and $r(\boldsymbol{\sigma})$ in analogous way.

The data $f$ and $g$ for our example, are chosen so that the exact solution is $u(x,y) = \dfrac{xy}{(x+1.05)^2 + y^2}$, and $\Omega := (-1,1)^2 \backslash [0,1]^2$. We notice that in this case $u$ has a singularity at $(-1.05, 0)$, which does not belong to $\Omega$, but it is very close to $\partial\Omega$. Then, $u$ has a numerical singularity in a neighborhood of $(-1,0) \in \Gamma$.

Then, the purpose of this example, is to show the performance of the following adaptive algorithm (cf. [22]). *Given an a posteriori error estimator* $\eta := \sum_{T\in\mathcal{T}_h} \eta_T^2$:

1. Start with a coarse mesh $\mathcal{T}_h$.
2. Solve the Galerkin scheme for the current mesh $\mathcal{T}_h$.
3. Compute $\eta_T$ for each triangle $T \in \mathcal{T}_h$.
4. Consider stopping criterion and decide to finish or go to the next step.
5. Apply *Blue-green* procedure to refine each element $T' \in \mathcal{T}_h$ such that
$$\eta_{T'} \geq \frac{1}{2} \max\{\eta_T : T \in \mathcal{T}_h\}.$$
6. Define the resulting mesh as the new $\mathcal{T}_h$ and go to step 2.

Table 1 reports the histories of convergence of the individual and total errors for a sequence of uniform and adaptive refinements, respectively. We notice that the adaptive refinement algorithm is able to recognize the numerical singularity, and then the induced sequence of adapted meshes let us to improve the quality of approximation, better than the corresponding when uniform refinement is performed. In addition, we observe that index of efficiency $e/\eta$ remains bounded, indicating that $\eta$ is reliable and efficient, despite the fact that $g$ in this case is not piecewise polynomial. Figure 1 displays some adapted meshes, generated by the proposed adaptive algorithm, from which we observe that the numerical singularity is detected.

Table 1: History of convergence of Example provided, considering uniform (up) and adaptive (bottom) refinements.

| $N$ | $e_0(u)$ | $r_0(u)$ | $e(\boldsymbol{\sigma})$ | $r(\boldsymbol{\sigma})$ | $e$ | $r$ | $\eta$ | $e/\eta$ |
|---|---|---|---|---|---|---|---|---|
| 34 | 4.6846e-01 | – | 4.2795e+00 | – | 4.3051e+00 | – | 806.54 | 0.0053 |
| 128 | 3.8157e-01 | 0.3095 | 4.5340e+00 | – | 4.5500e+00 | – | 434.93 | 0.0105 |
| 496 | 2.3974e-01 | 0.6862 | 5.1609e+00 | – | 5.1665e+00 | – | 213.32 | 0.0242 |
| 1952 | 1.1636e-01 | 1.0552 | 4.5542e+00 | 0.1826 | 4.5557e+00 | 0.1837 | 103.90 | 0.0438 |
| 7744 | 5.4734e-02 | 1.0947 | 4.5472e+00 | 0.0022 | 4.5475e+00 | 0.0026 | 50.360 | 0.0903 |
| 30848 | 2.0412e-02 | 1.4272 | 3.4610e+00 | 0.3950 | 3.4611e+00 | 0.3951 | 23.629 | 0.1465 |
| 123136 | 5.4906e-03 | 1.8972 | 1.8229e+00 | 0.9264 | 1.8229e+00 | 0.9264 | 11.484 | 0.1587 |
| 34 | 4.6846e-01 | – | 4.2795e+00 | – | 4.3051e+00 | – | 806.54 | 0.0053 |
| 99 | 3.8166e-01 | 0.3835 | 4.5365e+00 | – | 4.5526e+00 | – | 434.93 | 0.0105 |
| 192 | 2.3979e-01 | 1.4034 | 5.2105e+00 | – | 5.2160e+00 | – | 213.28 | 0.0245 |
| 289 | 1.1864e-01 | 3.4415 | 4.7172e+00 | 0.4865 | 4.7187e+00 | 0.4901 | 103.90 | 0.0454 |
| 386 | 5.9916e-02 | 4.7210 | 4.8647e+00 | – | 4.8651e+00 | – | 50.740 | 0.0959 |
| 483 | 3.6723e-02 | 4.3672 | 4.1212e+00 | 1.4798 | 4.1213e+00 | 1.4802 | 25.579 | 0.1611 |
| 580 | 3.3800e-02 | 0.9067 | 3.2165e+00 | 2.7084 | 3.2167e+00 | 2.7083 | 17.125 | 0.1878 |
| 1021 | 3.2498e-02 | 0.1389 | 2.4170e+00 | 1.0106 | 2.4172e+00 | 1.0105 | 11.726 | 0.2061 |
| 1747 | 2.2408e-02 | 1.3843 | 1.8179e+00 | 1.0606 | 1.8181e+00 | 1.0607 | 8.7937 | 0.2067 |
| 3633 | 2.2308e-02 | 0.0122 | 1.2888e+00 | 0.9395 | 1.2890e+00 | 0.9393 | 6.0359 | 0.2136 |
| 7208 | 6.5209e-03 | 3.5904 | 8.8348e-01 | 1.1024 | 8.8350e-01 | 1.1027 | 4.2152 | 0.2096 |
| 14979 | 3.9386e-03 | 1.3786 | 6.3307e-01 | 0.9113 | 6.3309e-01 | 0.9113 | 2.9475 | 0.2148 |
| 29052 | 1.7638e-03 | 2.4257 | 4.4686e-01 | 1.0517 | 4.4687e-01 | 1.0517 | 2.1120 | 0.2116 |
| 56844 | 1.1159e-03 | 1.3639 | 3.2513e-01 | 0.9476 | 3.2513e-01 | 0.9476 | 1.5154 | 0.2146 |
| 114267 | 4.5185e-04 | 2.5896 | 2.2625e-01 | 1.0386 | 2.2625e-01 | 1.0386 | 1.0765 | 0.2102 |
| 219394 | 2.8879e-04 | 1.3725 | 1.6567e-01 | 0.9555 | 1.6567e-01 | 0.9555 | 0.7781 | 0.2129 |

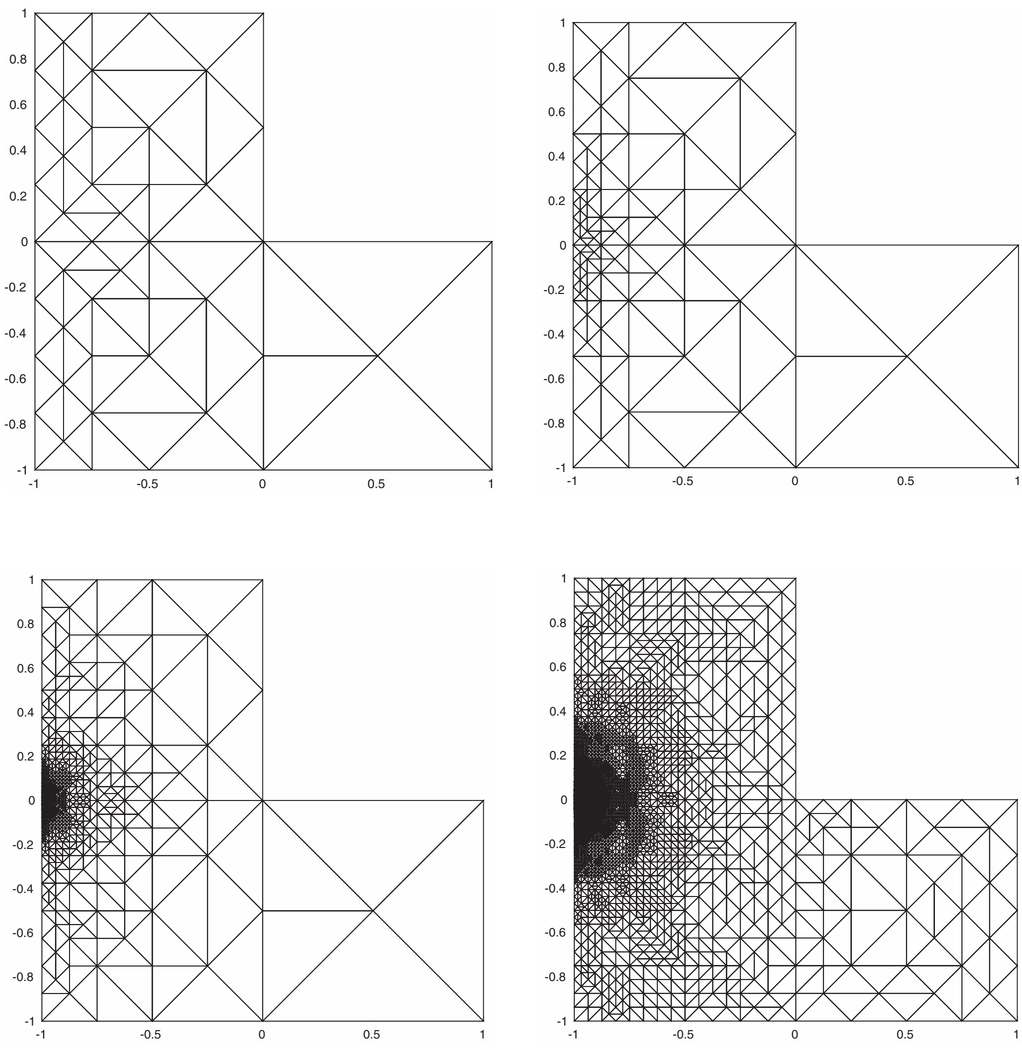


Figure 1: Adapted meshes corresponding (top-bottom, left-right) to 192, 386, 3633 and 29052 dofs, for Example considered, with Dirichlet boundary condition (based on $\eta$).

## CONCLUDING REMARKS

In this paper, we have developed an a posteriori error analysis for a dual mixed formulation of Poisson problem in the plane, with non homogeneous Dirichlet boundary condition. By establishing a new kind of *quasi-Helmholtz decomposition of functions in* $H(\mathrm{div};\Omega)$ (cf. Lemma 3.3), we are able to obtain an a posteriori error estimator, which consists of six residual terms, and results to be reliable and locally efficient with respect to the error measured in its *natural norm* on $H(\mathrm{div};\Omega)\times L^2(\Omega)$. In this sense, we have generalized the results obtained in previous works ( [1], [10], for example), and without invoking the so called *saturation assumption*.

The results of numerical experiment, included in this work, are in agreement with our theoretical analysis. Here, we notice that the estimator is able to help us to identify which part of the domain is localized the numerical singularity of the exact solution. As a consequence, the adaptive algorithm, based on this estimator, let us to improve the quality of the approximation.

Finally, since Lemma 3.3 can be proved for 3d case too, the current work can be extended to 3d, obtaining a reliable and locally efficient residual a posteriori error estimator, consisting also of six residual terms (cf. (3.31)).

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## REFERENCES

[1] A. Alonso. Error estimators for a mixed method. *Numer. Math.*, **74**(4) (1996), 385–395. doi:10.1007/s002110050222. URL `https://doi.org/10.1007/s002110050222`.

[2] T. Barrios, R. Bustinza, G.C. García & M. González. An a posteriori error estimator for a new stabilized formulation of the Brinkman problem. In “Numerical mathematics and advanced applications—ENUMATH 2013”, volume 103 of *Lect. Notes Comput. Sci. Eng.* Springer, Cham (2015), p. 253–261.

[3] T.P. Barrios, E.M. Behrens & R. Bustinza. A stabilized mixed method applied to Stokes system with nonhomogeneous source terms: the stationary case. Dedicated to Prof. R. Rodríguez, on the occasion of his 65th birthday. *Internat. J. Numer. Methods Fluids*, **92**(6) (2020), 509–527. doi:10.1002/fld.4793. URL `https://doi.org/10.1002/fld.4793`.

[4] T.P. Barrios, E.M. Behrens & M. González. Low cost a posteriori error estimators for an augmented mixed FEM in linear elasticity. *Appl. Numer. Math.*, **84** (2014), 46–65. doi:10.1016/j.apnum.2014.05.008. URL `https://doi.org/10.1016/j.apnum.2014.05.008`.

[5] T.P. Barrios, R. Bustinza, G.C. García & M. González. An a posteriori error analysis of a velocity-pseudostress formulation of the generalized Stokes problem. *J. Comput. Appl. Math.*, **357** (2019), 349–365. doi:10.1016/j.cam.2019.02.019. URL `https://doi.org/10.1016/j.cam.2019.02.019`.

[6] T.P. Barrios, J.M. Cascón & M. González. A posteriori error analysis of an augmented mixed finite element method for Darcy flow. *Comput. Methods Appl. Mech. Engrg.*, **283** (2015), 909–922. doi:10.1016/j.cma.2014.10.035. URL `https://doi.org/10.1016/j.cma.2014.10.035`.

[7] T.P. Barrios, J.M. Cascón & M. González. A posteriori error estimation of a stabilized mixed finite element method for Darcy flow. In "Boundary and interior layers, computational and asymptotic methods—BAIL 2014", volume 108 of *Lect. Notes Comput. Sci. Eng.* Springer, Cham (2015), p. 13–23.

[8] T.P. Barrios, J.M. Cascón & M. González. Augmented mixed finite element method for the Oseen problem: A priori and a posteriori error analysis. *Comput. Methods Appl. Mech. Engrg.*, **313** (2017), 216–238.

[9] T.P. Barrios & G.N. Gatica. An augmented mixed finite element method with Lagrange multipliers: A priori and a posteriori error analyses. *J. Comput. Appl. Math.*, **200**(2) (2007), 653–676. doi:10.1016/j.cam.2006.01.017. URL `https://doi.org/10.1016/j.cam.2006.01.017`.

[10] D. Braess & R. Verfürth. A posteriori error estimators for the Raviart-Thomas element. *SIAM J. Numer. Anal.*, **33**(6) (1996), 2431–2444. doi:10.1137/S0036142994264079. URL `https://doi.org/10.1137/S0036142994264079`.

[11] F. Brezzi & M. Fortin. "Mixed and hybrid finite element methods", volume 15 of *Springer Series in Computational Mathematics*. Springer-Verlag, New York (1991), x+350 p. doi:10.1007/978-1-4612-3172-1. URL `https://doi.org/10.1007/978-1-4612-3172-1`.

[12] C. Carstensen. A posteriori error estimate for the mixed finite element method. *Math. Comp.*, **66**(218) (1997), 465–476. doi:10.1090/S0025-5718-97-00837-5. URL `https://doi.org/10.1090/S0025-5718-97-00837-5`.

[13] C. Carstensen, S. Bartels & S. Jansche. A posteriori error estimates for nonconforming finite element methods. *Numer. Math.*, **92**(2) (2002), 233–256. doi:10.1007/s002110100378. URL `https://doi.org/10.1007/s002110100378`.

[14] P.G. Ciarlet. "The finite element method for elliptic problems", volume 40 of *Classics in Applied Mathematics*. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA (2002), xxviii+530 p. doi:10.1137/1.9780898719208. URL `https://doi.org/10.1137/1.9780898719208`. Reprint of the 1978 original [North-Holland, Amsterdam; MR0520174 (58 #25001)].

[15] P. Clément. Approximation by finite element functions using local regularization. *Rev. Française Automat. Informat. Recherche Opérationnelle Sér.*, **9**(R-2) (1975), 77–84.

[16] R.G. Durán. Mixed Finite Element Methods. In "Mixed finite elements, compatibility conditions, and applications", Lecture Notes in Mathematics. Springer-Verlag, Berlin; Fondazione C.I.M.E., Florence (2008), p. 1–44.

[17] G.N. Gatica & M. Maischak. A posteriori error estimates for the mixed finite element method with Lagrange multipliers. *Numer. Methods Partial Differential Equations*, **21**(3) (2005), 421–450. doi:10.1002/num.20050. URL `https://doi.org/10.1002/num.20050`.

[18] V. Girault & P.A. Raviart. "Finite element methods for Navier-Stokes equations", volume 5 of *Springer Series in Computational Mathematics*. Springer-Verlag, Berlin (1986), x+374 p. doi:10.1007/978-3-642-61623-5. URL `https://doi.org/10.1007/978-3-642-61623-5`. Theory and algorithms.

[19] R. Hiptmair. Finite elements in computational electromagnetism. *Acta Numer.*, **11** (2002), 237–339. doi:10.1017/S0962492902000041. URL `https://doi.org/10.1017/S0962492902000041`.

[20] J.E. Roberts & J.M. Thomas. Mixed and hybrid methods. In "Handbook of numerical analysis, Vol. II", Handb. Numer. Anal., II. North-Holland, Amsterdam (1991), p. 523–639.

[21] R. Verfürth. A posteriori error estimation and adaptive mesh-refinement techniques. *J. Comput. Appl. Math.*, **50**(1-3) (1994), 67–83. doi:10.1016/0377-0427(94)90290-9. URL `https://doi.org/10.1016/0377-0427(94)90290-9`.

[22] R. Verfürth. "A Review of a Posteriori Error Estimation and Adaptive Mesh-Refinement Techniques". Advances in Numerical Mathematics. Wiley-Teubner, Chichester (1996), xx+150 p.

[23] B.I. Wohlmuth & R.H.W. Hoppe. A comparison of a posteriori error estimators for mixed finite element discretizations by Raviart-Thomas elements. *Math. Comp.*, **68**(228) (1999), 1347–1378. doi:10.1090/S0025-5718-99-01125-4. URL `https://doi.org/10.1090/S0025-5718-99-01125-4`.