

## Theoretical and Numerical Aspects of a Third-order Three-point Nonhomogeneous Boundary Value Problem

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**ABSTRACT.** In this paper we are considering a third-order three-point equation with nonhomogeneous conditions in the boundary. Using Krasnoselskii's Theorem and Leray-Schauder Alternative we provide existence results of positive solutions for this problem. Nontrivial examples are given and a numerical method is introduced.

**Keywords:** numerical solutions, third-order, boundary value problem, Krasnoselskii's Theorem.

### 1 INTRODUCTION

Multi-point boundary value problems there has been attention of several studies mainly focused on the existence of solutions with qualitative and quantitative aspects, we recommend [2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 14, 15] and the references therein. It is well known that the Krasnoselskii's fixed point theorem, Avery-Peterson and Leggett-Williams theorems are massively used in this line.

In this paper, motivated by [13], we discuss the existence of a positive solution for the third-order boundary value problem:

$$u''' + f(t, u, u') = 0, \quad 0 < t < 1, \quad (1.1)$$

$$u(0) = u'(0) = 0, \quad u'(1) - \alpha u'(\eta) = \lambda, \quad (1.2)$$

where  $\eta \in (0, 1)$ ,  $\alpha \in [0, \frac{1}{\eta})$  are constants and  $\lambda \in (0, \infty)$  is a parameter. Essentially, we combine Leray-Schauder Alternative and Krasnoselskii's theorem to show the existence of a positive solution for (1.1)-(1.2) without supposing superlinearity on  $f$ . Numerical solutions are poorly

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explored, thus we complement this work presenting a numerical study for (1.1)-(1.2) based on Banach's Contraction Principle.

## 2 BACKGROUND MATERIAL

We begin this section by stating the following results.

**Theorem 1.** *Let  $E$  be a Banach space,  $C \subset E$  a closed and convex set,  $\Omega$  an open set in  $C$  and  $p \in \Omega$ . Then each completely continuous mapping  $T : \overline{\Omega} \rightarrow C$  has at least one of the following properties:*

(A1)  *$T$  has a fixed point in  $\overline{\Omega}$ .*

(A2) *There are  $u \in \partial\Omega$  and  $\lambda \in (0, 1)$  such that  $u = \lambda T(u) + (1 - \lambda)p$ .*

**Theorem 2.** *Let  $E$  be a Banach space and let  $K \subset E$  be a cone in  $E$ . Assume  $\Omega_1, \Omega_2$  are bounded open subsets of  $E$  with  $0 \in \Omega_1, \overline{\Omega}_1 \subset \Omega_2$ , and let  $T : K \cap (\overline{\Omega}_2 \setminus \Omega_1) \rightarrow K$  be a completely continuous operator such that, either*

(B1)  *$\|Tu\| \leq \|u\|, u \in K \cap \partial\Omega_1$ , and  $\|Tu\| \geq \|u\|, u \in K \cap \partial\Omega_2$ , or*

(B2)  *$\|Tu\| \geq \|u\|, u \in K \cap \partial\Omega_1$ , and  $\|Tu\| \leq \|u\|, u \in K \cap \partial\Omega_2$ .*

*Then  $T$  has a fixed point in  $K \cap (\overline{\Omega}_1 \setminus \Omega_1)$ .*

The first theorem is a well-known Leray-Schauder alternative and the second theorem is due to Krasnoselskii, see [1].

Let us set an auxiliary problem that will be useful in our context.

$$u''' + f(t, x, x') = 0, \quad 0 < t < 1, \quad (2.1)$$

$$u(0) = u'(0) = 0, \quad u'(1) - \alpha u'(\eta) = \lambda. \quad (2.2)$$

Related to this problem we have an important lemma.

**Lemma 3.** *Let  $x \in C^1[0, 1] := \{x \in C^1[0, 1], t \in [0, 1]\}$ , then we have a unique solution for (2.1)-(2.2). Moreover, this solution is expressed by*

$$u(t) = \int_0^1 G(t, s) f(s, x(s), x'(s)) ds + \frac{\alpha t^2}{2(1 - \alpha\eta)} \int_0^1 G_1(\eta, s) f(s, x(s), x'(s)) ds + \frac{\lambda t^2}{2(1 - \alpha\eta)}, \quad (2.3)$$

where  $G$  is the Green's function:

$$G(t, s) = \frac{1}{2} \begin{cases} (2t - t^2 - s)s, & s \leq t \\ (1 - s)t^2, & t \leq s \end{cases} \quad (2.4)$$

and

$$G_1(t, s) = \frac{\partial G(t, s)}{\partial t} = \begin{cases} (1-t)s, & s \leq t \\ (1-s)t, & t \leq s \end{cases}. \tag{2.5}$$

**Proof.** If  $u(t)$  is solution of (2.1), we can suppose that

$$u(t) = -\frac{1}{2} \int_0^t (t-s)^2 f(s, x, x') ds + At^2 + Bt + C.$$

From condition (2.2), we have  $B = C = 0$  and

$$A = \frac{1}{2(1-\alpha\eta)} \int_0^1 (1-s)f(s, x, x') ds - \frac{\alpha}{2(1-\alpha\eta)} \int_0^\eta (\eta-s)f(s, x, x') ds + \frac{\lambda}{(1-\alpha\eta)}$$

Thus (2.1)-(2.2) has a unique solution. Furthermore  $u(t) = -\frac{1}{2} \int_0^t (t-s)^2 f(s, x, x') ds +$

$$\begin{aligned} & \frac{t^2}{2(1-\alpha\eta)} \int_0^1 (1-s)f(s, x, x') ds \\ & - \frac{\alpha t^2}{2(1-\alpha\eta)} \int_0^\eta (\eta-s)f(s, x, x') ds + \frac{\lambda t^2}{2(1-\alpha\eta)} \\ & = -\frac{1}{2} \int_0^t (t-s)^2 f(s, x, x') ds + \frac{t^2}{2} \int_0^1 (1-s)f(s, x, x') ds \\ & + \frac{\alpha \eta t^2}{2(1-\alpha\eta)} \int_0^1 (1-s)f(s, x, x') ds - \frac{\alpha t^2}{2(1-\alpha\eta)} \int_0^\eta (\eta-s)f(s, x, x') ds + \frac{\lambda t^2}{2(1-\alpha\eta)} \\ & = \frac{1}{2} \int_0^t (-t^2 + 2st - s^2) f(s, x, x') ds + \frac{1}{2} \int_0^t (1-s)t^2 f(s, x, x') ds \\ & + \frac{1}{2} \int_t^1 (1-s)t^2 f(s, x, x') ds + \frac{\alpha t^2}{2(1-\alpha\eta)} \int_0^\eta (1-s)\eta f(s, x, x') ds \\ & + \frac{\alpha t^2}{2(1-\alpha\eta)} \int_\eta^1 (1-s)\eta f(s, x, x') ds - \frac{\alpha t^2}{2(1-\alpha\eta)} \int_0^\eta (\eta-s)f(s, x, x') ds + \frac{\lambda t^2}{2(1-\alpha\eta)} \\ & = \frac{1}{2} \int_0^t (2t - t^2 - s) s f(s, x, x') ds + \frac{1}{2} \int_t^1 (1-s)t^2 f(s, x, x') ds \\ & + \frac{\alpha t^2}{2(1-\alpha\eta)} \left( \int_0^\eta (1-\eta) s f(s, x, x') ds + \int_\eta^1 \eta (1-s) f(s, x, x') ds \right) + \frac{\lambda t^2}{2(1-\alpha\eta)} \\ & = \int_0^1 G(t, s) f(s, x, x') ds + \frac{\alpha t^2}{2(1-\alpha\eta)} \int_0^1 G_1(\eta, s) f(s, x, x') ds + \frac{\lambda t^2}{2(1-\alpha\eta)}. \end{aligned}$$

□

Defining  $x(t) = u(t)$  in Lemma 3 is easy to see that the solution of (1.1)-(1.2) can be expressed as fixed point of the operator  $T : C^1[0, 1] \rightarrow C^1[0, 1]$  defined by:

$$Tu(t) = \int_0^1 G(t, s) f(s, u, u') ds + \frac{\alpha t^2}{2(1-\alpha\eta)} \int_0^1 G_1(\eta, s) f(s, u, u') ds + \frac{\lambda t^2}{2(1-\alpha\eta)}. \tag{2.6}$$

**Remark 4.** Related to  $G$  and  $G_1$  we have useful properties that will be used in the next section.

- For all  $(t, s) \in [0, 1] \times [0, 1]$  :

$$0 \leq G_1(t, s) \leq (1-s)s$$

- For all  $(t, s) \in [0, 1] \times [0, 1]$ :

$$G(t, s) \leq G_1(1, s) = \frac{1}{2}(1-s)s$$

### 3 POSITIVE SOLUTIONS

Let  $E = \{u \in C^1[0, 1] : u(0) = 0\}$ , where  $C^1[0, 1]$  be the Banach space of continuously differentiable functions in  $[0, 1]$  equipped with

$$\|u\|_E = \max\{\|u\|_\infty, \|u'\|_\infty\}.$$

**Remark 1.** If  $u \in E$  then  $Tu$  satisfies  $Tu(0) = 0$ . Besides  $\|(Tu)'\|_\infty \geq \|Tu\|_E$ .

In order to prove the existence we need to consider some basic assumptions.

(H1) There exist positive constants  $A$ ,  $B$  and  $\beta$  such that

- $\max_{(s, v_1, v_2) \in [0, 1] \times [-\beta, \beta] \times [-\beta, \beta]} \{ |f(s, v_1, v_2)| \} \leq \frac{\beta(1-\alpha\eta)6B}{1+\alpha(1-\eta)}$
- $\lambda \leq A\beta(1-\alpha\eta)$
- $A+B \leq 1$ .

**Lemma 2.** Suppose that (H1) holds. Thus the problem (1.1)-(1.2) has a solution  $u^* \in E$  with  $\|u^*\|_E \leq \beta$ .

**Proof.** Let us consider the Theorem 1 with  $p = 0$  and  $\Omega = \{u \in E : \|u\|_E < \beta\}$ .

We claim that  $T$  is continuous and completely continuous. In fact, the continuity follows immediately from the Lebesgue dominated convergence theorem and noting that

$$\begin{aligned} |T(u)(t) - T(u_n)(t)| &\leq \int_0^1 G(t, s) |f(s, u(s), u'(s)) - f(s, u_n(s), u'_n(s))| ds + \\ &+ \left| \frac{\alpha t^2}{2(1-\alpha\eta)} \right| \int_0^1 G_1(\eta, s) |f(s, u(s), u'(s)) - f(s, u_n(s), u'_n(s))| ds \\ &\leq \int_0^1 G_1(1, s) |f(s, u(s), u'(s)) - f(s, u_n(s), u'_n(s))| ds + \end{aligned}$$

$$+ \left| \frac{\alpha}{2(1-\alpha\eta)} \right| \int_0^1 G_1(\eta, s) |f(s, u(s), u'(s)) - f(s, u_n(s), u'_n(s))| ds,$$

with  $u_n, u \in E$ . To show complete continuity we will use the Arzela-Ascoli's theorem. Let  $\Omega \subseteq E$  be bounded, in other words, there exists  $\Lambda_0 > 0$  with  $\|u\| \leq \Lambda_0$  for each  $u \in \Omega$ . Now if  $u \in \Omega$  we have

$$\begin{aligned} |(Tu)'(t)| &= \left| \int_0^1 G_1(t, s) f(s, u, u') ds + \frac{\alpha t}{1-\alpha\eta} \int_0^1 G_1(\eta, s) f(s, u, u') + \frac{\lambda t}{1-\alpha\eta} \right| \\ &\leq \int_0^1 |G_1(t, s) f(s, u, u')| ds + \frac{\alpha t}{1-\alpha\eta} \int_0^1 |G_1(\eta, s) f(s, u, u')| + \left| \frac{\lambda t}{1-\alpha\eta} \right| \\ &\leq \max_{t \in [0,1]} \frac{1-\alpha\eta + \alpha t}{1-\alpha\eta} \int_0^1 |(1-s)s f(s, u, u')| ds + \left| \frac{\lambda t}{1-\alpha\eta} \right| \\ &\leq \frac{1+\alpha(-\eta+1)}{1-\alpha\eta} \int_0^1 |(1-s)s| |f(s, u, u')| ds + \left| \frac{\lambda}{1-\alpha\eta} \right|. \end{aligned}$$

Then  $T\Omega$  is a bounded equicontinuous family on  $[0, 1]$ . Consequently the Arzela-Ascoli theorem implies  $T : E \rightarrow E$  is completely continuous.

In addition, suppose there are  $u \in \partial\Omega$  and  $\lambda \in (0, 1)$  with  $u(x) = \lambda Tu(x)$ . According (H1) we have:

$$\begin{aligned} \|Tu\|_E &< \|(Tu)'\|_\infty = \max_{t \in [0,1]} |(Tu)'(t)|, \\ &\leq \max_{t \in [0,1]} \frac{1+\alpha(-\eta+1)}{1-\alpha\eta} \int_0^1 |(1-s)s| |f(s, u, u')| ds + \left| \frac{\lambda}{1-\alpha\eta} \right| \\ &\leq \max_{(s,v_1,v_2) \in [0,1] \times [-\beta,\beta] \times [-\beta,\beta]} \frac{1+\alpha(-\eta+1)}{1-\alpha\eta} |f(s, v_1, v_2)| \int_0^1 (1-s)s ds + \left| \frac{\lambda}{1-\alpha\eta} \right| \\ &\leq \max_{(s,v_1,v_2) \in [0,1] \times [-\beta,\beta] \times [-\beta,\beta]} \frac{1+\alpha(-\eta+1)}{1-\alpha\eta} \frac{|f(s, v_1, v_2)|}{6} + \left| \frac{\lambda}{1-\alpha\eta} \right| \\ &\leq \frac{1}{1-\alpha\eta} \left[ \frac{1+\alpha(1-\eta)}{6} \max |f(s, v_1, v_2)| + \lambda \right] \\ &\leq \frac{1}{1-\alpha\eta} \left[ \frac{1+\alpha(1-\eta)}{6} \frac{\beta(1-\alpha\eta)6B}{1+\alpha(1-\eta)} + \lambda \right] \\ &\leq \frac{1}{1-\alpha\eta} [\beta(1-\alpha\eta)B + A\beta(1-\alpha\eta)] \\ &\leq \beta A + \beta B \leq \beta. \end{aligned}$$

Therefore,  $\|u\|_E < \beta$  and (A2) in Theorem 1 cannot occur. Thus (A1) holds and there is  $u^* \in E$  such that  $\|u^*\|_E \leq \beta$ . □

**Theorem 3.** Suppose that (H1) holds and  $f(s, u, v) \geq 0, \forall (s, u, v) \in [0, 1] \times [-\beta, \beta] \times [-\beta, \beta]$ . Then (1.1)-(1.2) has at least one positive solution  $u^* \in E$ .

**Proof.** We start the proof defining the cone  $K \subset E$  by

$$K = \{u \in E : u \geq 0, u(0) = 0, u'(0) = 0\}.$$

From (H1) and the definition of  $G$  and  $G_1$ , we have that  $T$  applies  $K$  in  $K$ . As seen in the last result,  $T$  is completely continuous.

We shall apply Theorem 2. Thus, we will define  $\Omega_1 = \{u \in E; \|u\|_E < \beta\}$ ,  $\Omega_2 = \{u \in E; \|u\|_E < \alpha\}$  and we will show that the following conditions are true for all  $u \in K$ :

- (a) if  $\|u\|_E = \alpha$  then  $\|Tu\|_E \leq \alpha$ ;
- (b) if  $\|u\|_E = \beta$  then  $\|Tu\|_E \geq \beta$ .

In fact, the demonstration of (a) is similar to the proof of the Lemma 2. To prove (b) is necessary to verify that there is  $\bar{\gamma} > 0$  with

$$\|Tu\|_E \geq \|u\|_E, \quad \forall u \in K \cap \partial\Omega_3,$$

where  $\Omega_3 = \{u \in E; \|u\|_E < \bar{\gamma}\}$ .

Let us assume that the inequality is false, that is, for every  $\bar{\gamma}$  such that  $\beta > \bar{\gamma} > 0$  there exists  $u \in E$  with  $\|u\|_E = \bar{\gamma}$  and  $\|Tu\|_E < \bar{\gamma}$ . Thus for all  $n \in \{1, 2, \dots\}$  with  $\frac{1}{n} < \alpha$ , we can find  $u_n \in K$  such that

$$\|u_n\|_E = \frac{1}{n} \text{ and } \|Tu_n\|_E < \frac{1}{n}.$$

Then  $\|u_n\| \rightarrow 0$  and  $\|Tu_n\| \rightarrow 0$ , when  $n \rightarrow \infty$ . Being  $T$  continuous, we have  $\|T0\|_E = 0$ . On the other hand, using (H1) and the definition of  $G$  and  $G_1$  we have

$$\begin{aligned} \|T0\|_\infty &\geq \max_{t \in [0,1]} \left\{ \frac{\lambda t^2}{2(1-\alpha n)} \right\}, \\ &\geq \frac{\lambda}{2(1-\alpha n)} > 0 \end{aligned}$$

which is a contradiction. Therefore we have the result. □

**Remark 4.** Note that the most important step in the proof of Theorem 3 is to impose conditions to conclude that 0 is not fixed point of  $T$ .

**Example 3.1.** Let us consider (1.1)-(1.2) with

$$\begin{aligned} f(t, u, v) &= \frac{1}{4}t + u^2 + v^2 \\ \eta &= \frac{1}{10}, \quad \alpha = \frac{1}{3}, \quad \lambda = \frac{1}{4} \end{aligned}$$

Choosing the constants

$$\beta = 10, \quad A = 0.54, \quad B = 0.45,$$

we can easily verify that in these conditions the hypotheses (H1) are satisfied.

**Example 3.2.** Let us define

$$f(t, u, v) = \frac{1}{4}t + \sin(u) + \frac{1}{4}\cos(v)$$

$$\eta = \frac{1}{9}, \alpha = \frac{1}{6}, \lambda = \frac{14}{10}$$

As before, choosing the constants

$$\beta = 2, A = 0.75, B = 0.2,$$

we can verify that (H1) is satisfied.

### 4 NUMERICAL SOLUTIONS

In this section we show the existence and uniqueness for (1.1)-(1.2) using Banach Fixed Point Theorem. This approach is classical but very important to define numerical methods for our problem. Let us consider the iterative sequence

$$u^{k+1} = T(u^k)$$

and the basic assumptions

$$(H2) |f(s, u, u') - f(s, v, v')| \leq A \max \{ |u(s) - v(s)|, |u'(s) - v'(s)| \};$$

$$(H3) \frac{-t^2 + t}{2} + \frac{\alpha t \eta (-\eta + 1)}{2(1 - \alpha \eta)} \leq \frac{1}{A}.$$

**Theorem 1.** Suppose that (H1), (H2) and (H3) are satisfied. Then (1.1)- (1.2) has a unique solution  $u$  with  $\|u\|_E \leq \beta$ . Moreover,  $u^{k+1} = T(u^k) \rightarrow u$ .

**Proof.** Let us consider  $u, v \in \Omega$  with  $\|u\|_E \leq \beta$  and  $\|v\|_E \leq \beta$ . Then

$$\begin{aligned} \|Tu - Tv\|_E &= \|(Tu - Tv)'\|_\infty \\ &= \left| \int_0^1 G_1(t, s)[f(s, u, u') - f(s, v, v')]ds + \frac{\alpha t}{1 - \alpha \eta} \int_0^1 G_1(t, s)[f(s, u, u') - f(s, v, v')]ds \right| \\ &\leq A \max_s \{ |u(s) - v(s)|, |u'(s) - v'(s)| \} \left( \int_0^1 G_1(t, s)ds + \frac{\alpha t}{1 - \alpha \eta} \int_0^1 G_1(\eta, s)ds \right) \\ &\leq A \max_s \{ |u(s) - v(s)|, |u'(s) - v'(s)| \} \left( \frac{-t^2 + t}{2} + \frac{\alpha t \eta (-\eta + 1)}{2(1 - \alpha \eta)} \right) \end{aligned}$$

Using (H3) we obtain

$$\begin{aligned} &\leq A \max_s \{ |u(s) - v(s)|, |u'(s) - v'(s)| \} \frac{1}{A} \\ &\leq \max_s \{ |u(s) - v(s)|, |u'(s) - v'(s)| \} = \|u - v\|_E \quad \square \end{aligned}$$

Motivated by the last result we can define Algorithm 1.

In sequence we are presenting some examples in order to establish the effectiveness of Algorithm 1. In tables,  $\epsilon_\mu^k$  denotes  $\|u^* - u^k\|_\infty$  where  $u^*$  is the exact solution,  $\epsilon^k$  denotes  $\|u^{k+1} - u^k\|_\infty$  and  $\bar{\epsilon}^k = \frac{\|u^{k+1} - u^k\|_\infty}{\|u^{k+1}\|_\infty}$ . Still, ‘‘It’’ denotes ‘‘iteration’’.

**Algorithm 1** Fixed-Point

- 
- 1: Define an uniformly distributed mesh  $\{x_j\}$  in  $[0, 1]$
  - 2: Define an initial approximation  $u_j^0 = u^0(x_j)$
  - 3: **for**  $k = 0, 1, 2, \dots$ , **do**
  - 4:   Compute  $u_j^k$  using finite differences
  - 5:   Compute  $u_j^{k+1}$  using  $u^{k+1} = T(u^k)$  and Trapezoidal Rule
  - 6:   Test the convergence
  - 7: **end for**
- 

**Example 4.1.** *In this example, we consider*

$$\begin{aligned} f(x, u, u') &= -u' \\ \eta &= \frac{\pi}{4}, \alpha = \frac{1}{10}, \lambda = 0.770760306689242 \end{aligned}$$

The analytical solution is  $u^*(x) = 1 - \cos(x)$ . The Table 1 contains results of application in Example 4.1.

We can make additional tests. From Theorem 3 we have a solution for Examples 3.1 and 3.2 but in both case, we do not know which they are. Let us apply Algorithm 1 in these problems. For this purpose, we can consider the condition

$$\frac{\|u^{k+1} - u^k\|}{\|u^{k+1}\|_\infty} < 10^{-4}$$

as stopping criterion for the algorithm. The results for these examples are presented in Table 2 and 3, respectively. The illustrations of these results are given in Figure 1 and 2.



Table 1: Algorithm 1 considering Example 4.1.

It	$\epsilon_u^k$	$\epsilon^k$	$\bar{\epsilon}^k$
1	0.104585227251908	0.355112466879952	1.000000000000000
2	0.072538564385106	0.032046662866802	0.082773878760819
3	0.069264937033799	0.003273627351307	0.008384612437847
4	0.068925441261629	0.000339495772170	0.000868781674432
5	0.068890166416009	0.000035274845620	0.000090261428243

Table 2: Algorithm 1 considering Example 3.1.

It	$\epsilon_u^k$	$\epsilon^k$	$\bar{\epsilon}^k$
1	-	0.168278823890335	1.000000000000000
2	-	0.007563461402919	0.043012756518181
3	-	0.000744660869474	0.004216964422657
4	-	0.000077049209989	0.000436134198292

Table 3: Algorithm 1 considering Example 3.2.

It	$\epsilon_u^k$	$\epsilon^k$	$\bar{\epsilon}^k$
1	-	0.740971458506793	1.000000000000000
2	-	0.010141509530254	0.013876702158219
3	-	0.000276799190473	0.000378602975785
4	-	0.000007307338952	0.000009995000056

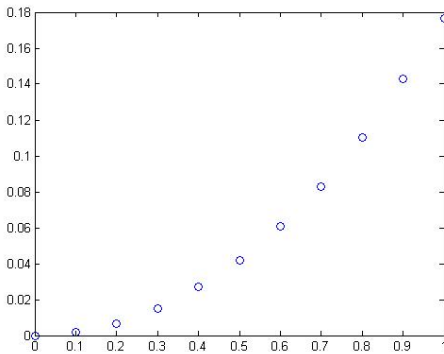


Figure 1: Numerical solution obtained from Example 1 using Algorithm 1.

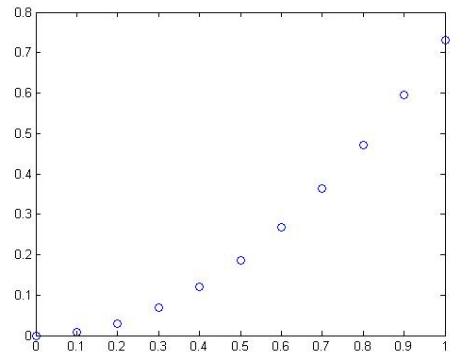


Figure 2: Numerical solution obtained from Example 2 using Algorithm 1.

**RESUMO.** Neste artigo, consideramos uma equação com três pontos de fronteira de terceira ordem com condições de contorno não homogêneas. Com uso do Teorema de Krasnoselskii e da Alternativa de Leray-Schauder, apresentamos resultados de existência para soluções positivas. Exemplos não triviais são fornecidos e um método numérico é introduzido.

**Palavras-chave:** soluções numéricas, terceira-ordem, problema de valor de contorno, Teorema de Krasnoselskii.

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