

Characterizing Block Graphs in Terms of One-vertex Extensions

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ABSTRACT. Block graphs has been extensively studied for many decades. In this paper we present a new characterization of the class in terms of one-vertex extensions. To this purpose, a specific representation based on the concept of boundary cliques is presented, bringing about some properties of the graph.

Keywords: block graph, one-vertex-extension.

1 INTRODUCTION

Block graphs have been extensively studied for many decades with characterizations based on different approaches since the first one in 1963 until today. In this paper we present a new characterization of the class in terms of one-vertex extensions. As block graphs are a subclass of chordal graphs, properties of this class can be successfully particularized: a specific representation of block graphs based on the concept of boundary cliques is presented, bringing about some properties of the graph.

Harary [7] introduced the definition of a block graph based on structural properties and presented a classical characterization: the block graph $B(G)$ of a given graph G is that graph whose vertices are the blocks (maximal 2-connected components) B_1, \dots, B_k of G and whose edges are determined by taking two vertices B_i and B_j as adjacent if and only if they contain a cut-vertex (its removal disconnects the graph) of G in common. A graph is called a *block graph* if it is the block graph of some graph.

Characterization 1. [7] *A graph is a block graph if and only if all its blocks are complete.*

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Howorka [8], Bandelt and Mulder [2] and Behtoei *et al.* [3] presented characterizations based on metric conditions.

Characterization 2. [8] *A graph is a block graph if and only if for every four vertices u, v, w, x , the larger two of the distance sums $d(u, v) + d(w, x)$, $d(u, w) + d(v, x)$, $d(u, x) + d(v, w)$ are equal.*

Characterization 3. [2] *A graph $G = (V, E)$ is a block graph if and only if for every vertices $u, v, w \in V$, G satisfies*

1. $I(u, v) \subseteq I(u, w) \cup I(v, w)$
2. $I(u, w) \subseteq I(u, v) \cup I(v, w)$
3. $I(v, w) \subseteq I(u, v) \cup I(u, w)$

where $I(x, y) = \{z \in V; d(x, y) = d(x, z) + d(z, y)\}$, $x, y \in V$.

Characterization 4. [3] *A graph G is a block graph if and only if it satisfies:*

1. *the shortest path between any two vertices of G is unique and*
2. *for each edge $e = uv \in E$, if $x \in N_e(u)$ and $y \in N_e(v)$, then, and only then, the shortest path between x and y contains the edge e , where $N_{e=uv}(v) = \{w \in V; d(v, w) < d(u, w)\}$.*

Bandelt and Mulder [1] presented a characterization based on forbidden subgraphs.

Characterization 5. [1] *A graph is a block graph if and only if it is $C_{n \geq 4}$ -free and diamond-free.*

Mulder and Nebeský [11] characterized block graphs using an algebraic approach, a binary operation $+$ (*leap operation*) on a finite nonempty set V such that for $u, v, w \in V$,

1. $(u + v) + u = u$.
2. if $u \neq v$ then $(u + v) + v \neq u$.
3. if $u + v \neq v$ then $((u + v) + v) + u \neq u$.
4. if $u \neq v = u + v$, $u + w \neq v$ and $v + w \neq u$ then $u + w = v + w$.

The underlying graph $G_+ = (V, E_+)$ of $+$ is such that $uv \in E_+$ if and only if $u \neq v$, $u + v = v$ and $v + u = u$.

Intuitively, for any two vertices u and w in different blocks, the leap operation produces the cut-vertex z in the block of u on the way to w , i.e., $u + w = z$. If u and w are in the same block, then $u + w = w$.

Characterization 6. [11] *G is a block graph if and only if it is the underlying graph of a leap operation on V .*

Recently, the subject was resumed. Dress *et al.* [5] characterized block graphs in terms of their vertex-induced partitions: any partition of a given finite set V is a V -partition, and $P_V = (p_v)_{v \in V}$ is a V -indexed family of V -partitions. A family P_V is a compatible family of V -partitions if, for any two distinct elements $u, v \in V$, the union of the set in p_v that contains u and the set in p_u that contains v coincides with the set V , and $\{v\} \in p_v$ holds for all $v \in V$. Let $P(V)$ denote the set of all compatible families of V -partitions.

Characterization 7. [5] *There is a bijective function between the block graphs with vertex set V and $P(V)$.*

Mulder [10] presented a surprisingly simple characterization.

Characterization 8. [10] *The graph G is a block graph if and only if there exists a unique induced path between any two vertices in G .*

2 BACKGROUND

Basic concepts about chordal graphs are assumed to be known and can be found in Blair and Peyton [4] and Golubic [6]. In this section, the most pertinent concepts are reviewed.

Let $G = (V(G), E(G))$, or simply $G = (V, E)$, be a connected graph, where $|E| = m$ and $|V| = n$. The set of neighbors of a vertex $v \in V$ is denoted by $N_G(v) = \{w \in V; vw \in E\}$ and its closed neighborhood by $N_G[v] = N_G(v) \cup \{v\}$. Two vertices u and v are *true twins* in G if $N_G[u] = N_G[v]$ and *false twins* in G if $N_G(u) = N_G(v)$. For any $S \subseteq V$, the subgraph of G induced by S is denoted $G[S]$. The set S is a *clique* if $G[S]$ is complete. A vertex $v \in V$ is said to be *simplicial* in G when $N_G(v)$ is a *clique* in G .

It is worth mentioning two kinds of cliques in a chordal graph G . A *simplicial clique* is a maximal clique containing at least one simplicial vertex. A simplicial clique Q is called a *boundary clique* if there exists a maximal clique Q' , $Q \neq Q'$, such that $Q \setminus Q'$ is a set of simplicial vertices of G .

A *perfect elimination ordering* (*peo*) of a graph $G = (V, E)$ is a bijective function $\sigma : \{1, \dots, n\} \rightarrow V$ such that $\sigma(i)$ is a simplicial vertex in the induced subgraph $G_i = G[\{\sigma(i), \dots, \sigma(n)\}]$, for $1 \leq i < n$. A *peo* is ultimately an arrangement of V in a sequence $\sigma(V) = [\sigma(1), \dots, \sigma(n)]$. It is well known that a graph G is chordal if and only if G admits a perfect elimination ordering.

3 BOUNDARY REPRESENTATION

In this section we present a representation of block graphs based on the concept of a perfect elimination ordering of the graph. As in a *peo*, where a vertex is eliminated when it is simplicial in the remaining graph, in this proposed representation, a maximal clique is eliminated when it is a boundary clique in the remaining graph. As all elements of the maximal clique are stored, the graph can be easily recovered. The representation is defined as follows; its structure is similar to the one presented in [9].

Property 3. *The sequence provided by all vertices of P_1 , followed by all vertices of P_2 , and so on, up to P_ℓ is a perfect elimination ordering of G . Observe that, since there is no order in the set P_i , $i = 1, \dots, \ell$, several sequences can be built.*

Employing the algorithm for the graph in Figure 1 we have:

$$BR(G) = [(\{a\}, b), (\{c\}, d), (\{e, f\}, d), (\{k\}, g), (\{j\}, i), (\{o, p\}, m), (\{b\}, d), (\{m, n, \ell\}, h), (\{d, h, g, i\}, \emptyset)]$$

4 ONE-VERTEX EXTENSIONS

The concept of *one-vertex extension* was introduced by Bandelt and Mulder [1].

Let $G = (V, E)$ be a graph, $v \in V$ and $u \notin V$. An extension of G to a graph $G' = (V', E')$ is a *one-vertex extension* if it obeys one of the following three rules:

- (α) $V' = V \cup \{u\}$ and $E' = E \cup \{vu\}$, i.e., $N_{G'}(u) = \{v\}$ (u is a pendant vertex).
- (β) $V' = V \cup \{u\}$ and $E' = E \cup \{xu; x \in N_G[v]\}$, i.e., $N_{G'}[u] = N_G[v]$ (u is a true twin of v).
- (γ) $V' = V \cup \{u\}$ and $E' = E \cup \{xu; x \in N_G(v)\}$, i.e., $N_{G'}(u) = N_G(v)$ (u is a false twin of v).

The special cases of (α), (β) and (γ) restricted to a simplicial vertex $v \in V$ are denoted by (α^*), (β^*) and (γ^*), respectively.

In order to generate a graph $G = (V, E)$, it is possible to establish a building sequence. A *one-extension sequence (oes)* of G is a sequence of triples

$$\Pi(G) = [\pi(1), \dots, \pi(n)]$$

being $\pi(i) = (e_i, v_i, u_i)$, $i = 2, \dots, n$, such that

1. $e_i \in \{(\alpha), (\beta), (\gamma), (\alpha^*), (\beta^*), (\gamma^*)\}$;
2. $v_i = u_j$, for some $j < i$;
3. $u_i \neq u_k$, $1 \leq k \leq i - 1$;

and $\pi(1)$ is the special initial triple $(\emptyset, \emptyset, u_1)$.

Bandelt and Mulder [1] presented characterizations of distance hereditary graphs and ptolemaic graphs; the first one using the extensions (α), (β) and (γ), and the second one using (α), (β) and (γ^*). Theorem 4.1, presented below, shows a characterization of block graphs using one-vertex extensions.

Consider a graph G , $CV(G)$ the set of cut-vertices, $Simp(G)$ the set of simplicial vertices and $\mathbb{Q}(G)$ the set of maximal cliques of the graph.

Theorem 4.1. *A graph $G = (V, E)$ is a block graph if and only if there is a sequence $\Pi(G)$ of G composed by type (α) and type (β^*) extensions.*

Proof. Consider a block graph G with ℓ maximal cliques and its boundary representation $BR(G) = [(P_1, s_1), \dots, (P_\ell, \emptyset)]$. It is possible to construct a sequence $\Pi(G)$ by transversing the boundary representation in reverse order.

Let (P_ℓ, \emptyset) , $v \in P_\ell$ and $\pi(1) = (\emptyset, \emptyset, v)$. Consider $w \in P_\ell \setminus \{v\}$ and $\pi(2) = (\alpha, v, w)$. For $x \in P_\ell \setminus \{v, w\}$, let be the triple (β^*, w, x) . So, there are the following elements of the sequence $\Pi(G)$: $\pi(i) = (\beta^*, w, x)$, $i = 3, \dots, |P_\ell|$. Thus, a first maximal clique of G is obtained.

Let $(P_{\ell-j}, s_{\ell-j})$, $j = 1, \dots, \ell - 1$. By the definition of boundary representation, $s_{\ell-j} \in Q_k$, $\ell - j + 1 \leq k \leq \ell$. Consider $v \in P_{\ell-j}$. The graph obtained by the extension $\pi(1 + \sum_{k=\ell-j+1}^{\ell} |P_k|) = (\alpha, s_{\ell-j}, v)$ has v as a pendant vertex and $s_{\ell-j}$ as a cut-vertex. The vertices v and $s_{\ell-j}$ belong to a new maximal clique Q . For $x \in P_{\ell-j} \setminus \{v\}$, let be the triple (β^*, v, x) . Thus, there are the following elements of the sequence $\Pi(G)$: $\pi(i) = (\beta^*, v, x)$, $i = 2 + \sum_{k=\ell-j+1}^{\ell} |P_k|, \dots, \sum_{k=\ell-j}^{\ell} |P_k|$. These extensions increase the clique Q to which vertex v belongs in G . Then, we obtain the one-extension sequence of G , $\Pi(G)$, composed by type (α) and type (β^*) extensions.

Conversely, consider $\Pi(G) = [(\emptyset, \emptyset, v)]$. The resulting graph G is a trivial graph $(\{v\}, \emptyset)$ and it is a block graph.

Consider $H = (V(H), E(H))$ a block graph with $n - 1$ vertices obtained by $\Pi(H) = [\pi(1), \dots, \pi(n - 1)]$ a sequence of (α) and (β^*) extensions. Let $v \in V(H)$, Q the maximal clique to which it belongs in H and $u \notin V(H)$.

Let $\Pi(G) = \Pi(H) \parallel \pi(n) = [\pi(1), \dots, \pi(n - 1), \pi(n)]$.

If $\pi(n) = (\alpha, v, u)$, two cases must be analyzed.

1. v is a simplicial vertex of H . Then, $CV(G) = CV(H) \cup \{v\}$ and $Simp(G) = (Simp(H) \setminus \{v\}) \cup \{u\}$.
2. v is a cut-vertex of H . Then, $CV(G) = CV(H)$ and $Simp(G) = Simp(H) \cup \{u\}$.

In both cases, the set of maximal cliques $\mathbb{Q}(G) = \mathbb{Q}(H) \cup \{vu\}$.

If $\pi(n) = (\beta^*, v, u)$, v must be a simplicial vertex in H . So, $CV(G) = CV(H)$, $Simp(G) = Simp(H) \cup \{u\}$ and $\mathbb{Q}(G) = (\mathbb{Q}(H) \setminus Q) \cup \{Q'\}$ where Q is a maximal clique such that $v \in Q$ and $Q' = Q \cup \{u\}$.

In any case, G is a block graph. □

The proof of Theorem 4.1 provides a possible one-extension sequence of a block graph. As an example, consider the block graph G in Figure 1 and the boundary representation of the same graph presented in Section 3. The one-extension sequence obtained from $BR(G)$ is

$$\begin{aligned} \Pi(G) = [& (\emptyset, \emptyset, d), (\alpha, d, h), (\beta^*, h, g), (\beta^*, h, i), (\alpha, h, m), (\beta^*, m, n), \\ & (\beta^*, m, \ell), (\alpha, d, b), (\alpha, m, o), (\beta^*, o, p), (\alpha, i, j), (\alpha, g, k), \\ & (\alpha, d, e), (\beta^*, e, f), (\alpha, d, c), (\alpha, b, a)]. \end{aligned}$$

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RESUMO. Grafos bloco têm sido extensivamente estudados por muitas décadas. Neste artigo apresentamos uma nova caracterização da classe em termos de extensões em um vértice. Com esse objetivo, é definida uma representação especial, baseada no conceito de cliques limítrofes, ressaltando propriedades dos grafos bloco.

Palavras-chave: grafo bloco, extensão em um vértice.

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